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COMPOSITIONS, BIJECTIONS, AND ENUMERATIONS

by

CHARLES R. DEDRICKSON III

(Under the Direction of Hua Wang)

ABSTRACT

In this thesis we give an introduction to colored-compositions of an integer. This is a generalization of traditional integer compositions, and we show a few results for n-color compositions which are analogous to regular compositions with both combinatorial and analytic proofs. We also show several bijections between various types of compositions to certain types of numeric strings, and provide a generalization of a classic bijection between compositions and binary strings.

Key Words: composition, bijection, generating function, n-colored

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CHARLES R. DEDRICKSON III

B.S. in Mathematics

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment

of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2012

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CHARLES R. DEDRICKSON III

Major Professor: Hua Wang

Committee: Andrew Sills

David Stone

Electronic Version Approved:

December 2012

ACKNOWLEDGMENTS

I am in debt to many people for this thesis. I wish to express my gratitude to my professors and fellow students of the department of mathematical sciences. In particular, I greatly appreciate the help I received from my thesis committee. Last but not least, I want to thank my family for their consistent support.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In this chapter, we introduce the concept of integer composition, and the generalization to n-color compositions. We also show some classic results in the study of compositions, and some more recent results about n-color compositions.

1.1 Integer Compositions

A *composition* of a positive integer n is a finite sequence of positive integers

$$\{n_1, n_2, n_3, \ldots, n_k\}$$

such that

$$n_1 + n_2 + n_3 + \cdots + n_k = n.$$

We often write the composition as

$$n_1 + n_2 + n_3 + \cdots + n_k$$

and call the summands *parts.* For instance, the 8 compositions of 4 are listed below:

$$4,\ 3+1,\ 2+2,\ 1+3,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 1+1+1+1.$$

The following theorem was first published in [11].

The number of integer compositions of n into k parts is denoted by $C_k(n)$ and the total number of compositions of n is denoted by $C(n)$.

Theorem 1.1.1. *The number of compositions of an integer n into k parts is $\binom{n-1}{k-1}$. The total number of compositions of a positive integer n is 2^{n-1}.*

Proof. Consider

$$\underbrace{\{\, 1 \square 1 \square \cdots \square 1 \square 1 \,\}}_{n\ 1\text{'s}}.$$

By placing a "+" into $k-1$ of the $n-1$ boxes above, and a "," in the remaining boxes, we obtain a composition of n with exactly k parts. Clearly, there are $\binom{n-1}{k-1}$ ways to do this, each producing a distinct composition. Now consider a given composition of n with k parts. This uniquely determines the location of the pluses or commas. Since the total number of compositions of n is the number of compositions with $1,\ 2,\ 3,\ \ldots,$ or n parts we have

$$C(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

$\square$

Example 1.1.2.

$$\{2, 3, 1, 2\}$$

is a composition of $n = 8$ *with* $k = 4$ *parts because*

$$2 + 3 + 1 + 2 = 8.$$

This composition corresponds to

$$\{1 + 1, 1 + 1 + 1, 1, 1 + 1\}.$$

Notice the 8 1*'s,* $4 - 1 = 3$ *commas, and* 4 *plus signs. By swapping the position of any comma with any plus sign we have a different composition of* 8 *with* 4 *parts.*

A composition can be naturally represented as tiling a $1 \times n$ board with tiles of dimension $1 \times n_i$ $(1 \leq i \leq k)$. For example, the composition $2 + 3 + 1 + 2 = 8$ can be represented as

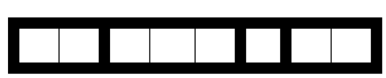


Figure 1.1: Tiling representation of $2 + 3 + 1 + 2 = 8$.

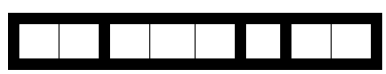
A classic bijection maps a composition of n to a binary word of length $n-1$. See, for instance, [7, 12]. Using the tiling representation of a composition to guide us, we can consider the vertical line segments separating the tiles (which we refer to as the internal vertical lines because we exclude the farthest left and right edges) in Figure 1.1 and map a vertical line segment to a 0 if it is bold and otherwise to a 1. The above composition of 8 is then mapped to a binary word

$$1011001 \tag{1.1}$$

of length 7. This leads to a natural definition of the *conjugate* of a composition. By switching each 1 to a 0 and each 0 to a 1 and mapping back to a composition, we obtain the conjugate of a composition.

Example 1.1.3. *Let us find the conjugate of* $2+3+1+2$.

First examine the tiling:



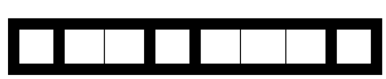
Then generate its corresponding binary string:

$$1011001.$$

We then conjugate the string to obtain:

$$0100110$$

Finding this string's corresponding tiling, we have:



And from this we obtain the composition we are looking for.

$$1+2+1+3+1$$

is the conjugate of

$$2+3+1+2.$$

1.2 n-Color Compositions

An *n-color composition* of a positive integer v is a composition in which each part of size n is one of n different colors [1]. Note that n refers to the size of a given part, not the size of the composition. We denote the different colors by a subscript (i.e. n_i denotes a part n with color i, where $1 \le i \le n$). For instance, in the composition of 6, $2+1+3$, the part 2 has two possible colors, the part 1 has only one color, the part 3 has three possible colors. All the n-color compositions with parts of the same size (in the same order) are shown below:

$$2_1+1_1+3_1, 2_1+1_1+3_2, 2_1+1_1+3_3, 2_2+1_1+3_1, 2_2+1_1+3_2, 2_2+1_1+3_3. \quad (1.2)$$

The number of n-color compositions of an integer v is denoted by $CC(v)$. We can see from the following example that $CC(2) = 3$ and $CC(3) = 8$.

Example 1.2.1.

$$\{1_1,\ 1_1\},\ \{2_1\},\ \textit{and}\ \{2_2\}$$

are the 3 *n-colored compositions of* 2. *With our other notation for compositions, these three compositions are*

$$1_1+1_1,\ 2_1,\ \textit{and}\ 2_2.$$

Similarly, the n*-color compositions of* 3 *are*

$$\begin{array}{cc} 3_1 & 2_2 + 1_1 \\ 3_2 & 1_1 + 2_1 \\ 3_3 & 1 + 2_2 \\ 2_1 + 1_1 & 1_1 + 1_1 + 1_1. \end{array}$$

The tiling representation can be naturally extended for n-color compositions by marking one of the k squares of a $1 \times k$ tile with a dot, called the "spotted tiling" [9]. For example, the n-color composition $2_2 + 3_2 + 1_1 + 2_1 = 8$ can be represented by Figure 1.2.

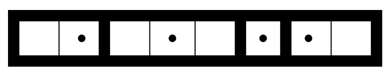


Figure 1.2: Tiling representation of $2_2 + 3_2 + 1_1 + 2_1 = 8$.

Many interesting studies and questions followed from the introduction of the n-color compositions. See, for instance, [2, 3, 6, 8] and the reference therein for some related topics. When introducing this concept, Agarwal [1] also asked for the analogue of MacMahon's zig-zag graph [10] as the conjugation of a composition. It was shown in [1] that the number of n-color compositions of v is F_{2v}, i.e. bisection of Fibonacci sequence (A001906 in [13]). In [9], an interesting bijection was established between the n-color compositions of v and compositions of $2v$ with odd parts. It is not hard to find other objects counted by F_{2v}. For instance, the "sum of the products of the elements in the compositions of v" is also counted by this sequence [13]. From (1.2) one can easily see the relation between this sum and the number of n-color compositions. Indeed, the number of choices of color schemes of a composition with given parts in a given order is exactly the product of the sizes of the parts.

In view of the aforementioned classical one-to-one correspondence between the compositions and the binary words, it is interesting to see that the "number of binary words with exactly $v-1$ strictly increasing runs" (A119900 in [13]) is also F_{2v}. Clearly each "strictly increasing run" is either a 0, 1, or 01. As an example, the binary word "100111101011100" can be written as

$$|1|0|01|1|1|1|01|01|1|1|0|0|$$

where strictly increasing runs are separated by $|$'s.

In Section 2.1, we will see a bijection between the n-color compositions and such binary words, offering a combinatorial proof for the following fact.

Proposition 1.2.2. *The number of n-color compositions of v is the same as the number of binary words with exactly $v-1$ strictly increasing runs.*

A similar bijection will also show that

Proposition 1.2.3. *The number of n-color compositions of v is the same as the number of 12-avoiding $\{0,1,2\}$ strings of length $v-1$.*

The second set of objects in Proposition 1.2.3 was counted by Katugampola in 2008 (A001906 in [13]). In fact, we will see that a one-to-one correspondence between the binary words and 12-avoiding strings follows naturally.

Proposition 1.2.2 provides a generalization of the bijection between compositions and binary words. And indeed from any binary string, one can generate a corresponding n-color composition by using the inverse of our map. We comment on the potential applications of these results in Section 2.4.

CHAPTER 2

SOME INTERESTING BIJECTIONS

This chapter is based on joint work with Alex Collins and Hua Wang [4]. We start with the bijection between the n-color compositions of v and binary words with $v-1$ strictly increasing runs. The establishment of this bijection not only generalizes the classical bijection between compositions and binary words, but also shows some potential in dealing with the open question "What is the conjugate of an n-color composition?" and obtaining bijections between different types of restricted n-color compositions.

2.1 Between n-Color Compositions and Binary Words

The bijection is displayed here in an algorithmic process. Given the spotted tiling representation of an n-color composition, we start from the left end and consider the left most tile. In every step (except for the last step) we decrease the number of "squares" by 1 and generate a strictly increasing run. This is described below and illustrated by Figure 2.1.

- If the left most part is 1_1 (i.e. a tile with one square that is marked), we remove this tile from the tiling and generate $|01|$ for the binary word;
- If the left most part is k_i for some $k > 1$ and $1 < i \leq k$ (i.e. the left most tile is of size > 1 and the marked square is not the first (left most) one), we remove the first square from the tiling (hence the new first part will be $(k-1)_{i-1}$) and generate $|1|$ for the binary word;
- If the left most part is k_1 for some $k > 1$ (i.e. the left most tile is of size > 1 and the marked square is the first one), we remove the first square and mark

the next one (hence the new first part will be $(k-1)_1$), generating $|0|$ for the binary word.

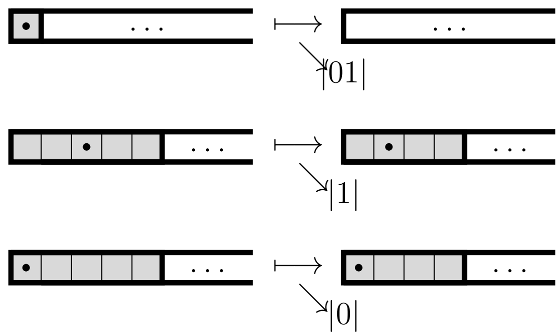


Figure 2.1: Generate a binary word from an n-color composition.

Note that a binary word generated this way will never have $|0|$ followed by a $|1|$ (in which case the two strictly increasing runs will be combined to count as one). Following this process, we will always have a tile of size 1 with the square marked in the end. We simply ignore this last square. This is shown by Figure 2.2, when there are two squares left before the last step.

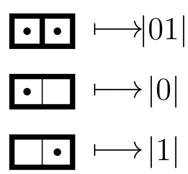


Figure 2.2: Three possibilities for the end of the process.

It is not difficult to see that the operations defined this way map the tiling representation of an n-color composition of v to a binary word with $v-1$ strictly increasing runs and this map is one-to-one. For example, the n-color composition

$$3_2 + 4_4 + 1_1 + 5_3 = 13$$

has a tiling representation as in Figure 2.3.

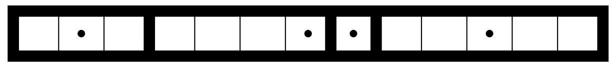


Figure 2.3: Tiling representation of $3_2 + 4_4 + 1_1 + 5_3 = 13$.

Under the operations defined above, this representation is mapped to the binary word

$$|1|0|01|1|1|1|01|01|1|1|0|0|. \tag{2.1}$$

Instead of providing a formal proof for this bijection, we show in detail how one can reverse the process and achieve an n-color composition of 13 from (2.1).

Starting from the end of the binary word (the right side), we have $|0|$ corresponding to the second operation in Figure 2.2. Then we have another $|0|$ corresponding to the third operation in Figure 2.1 and the process continues this way. This is illustrated in Figure 2.4.

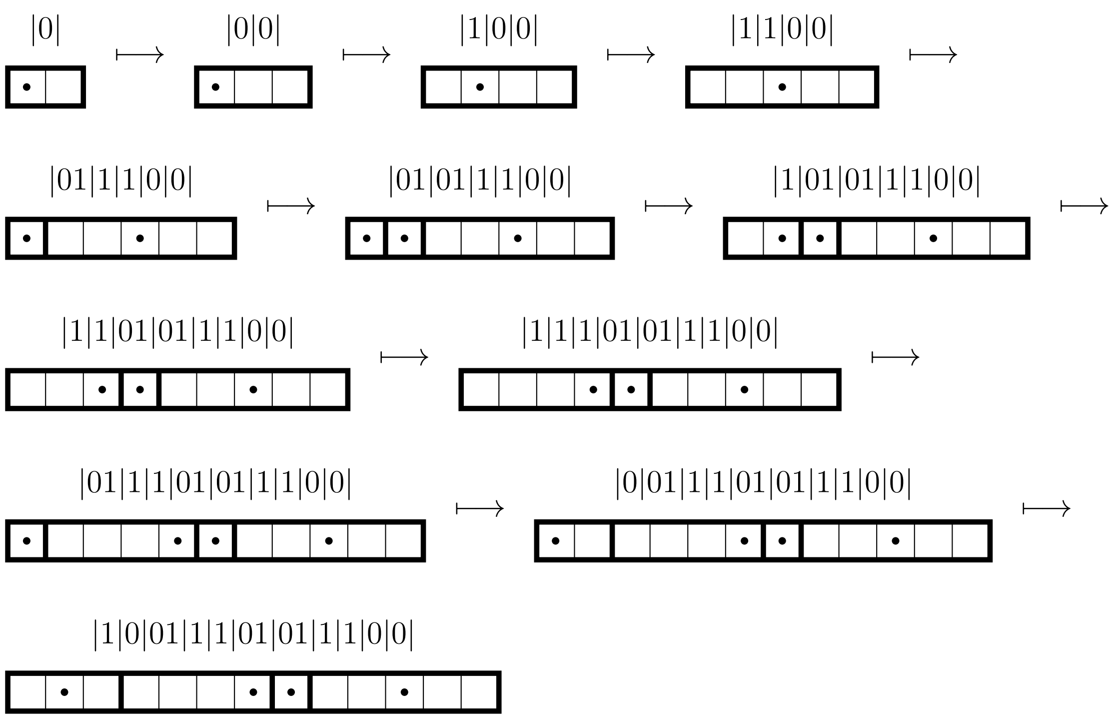


Figure 2.4: Generating an n-color composition from a binary word.

2.2 Between n-Color Compositions and the 12-Avoiding Strings

The process is similar to that in the previous subsection, illustrated in Figure 2.5. Note that any string generated from an n-color composition will avoid the pattern "12".

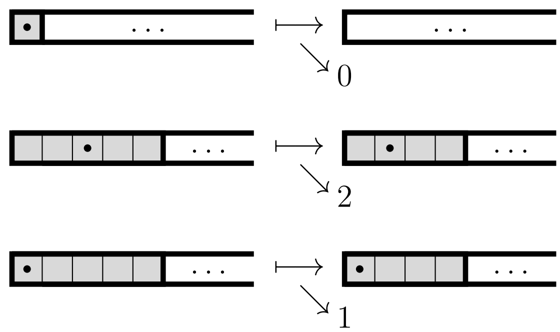


Figure 2.5: Generating a 12-avoiding string from an n-color composition.

Similar to before, we ignore the very end of the tiling. Again, Figure 2.6 shows the situation when there are two squares left.

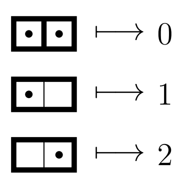


Figure 2.6: Three possibilities of the end of the process.

2.3 Between the Binary Words and the 12-Avoiding Strings

It is not hard to notice the similarity between the two aforementioned bijections. Indeed, from them follows a natural bijection between the binary words with exactly v strictly increasing runs (where $|0|1|$ is "avoided") and the $\{0, 1, 2\}$ strings of length v that avoids 12 through the following:

- $$0 \rightleftharpoons |01|;$$
- $$1 \rightleftharpoons |0|;$$
- $$2 \rightleftharpoons |1|.$$

2.4 Some Remarks

In this chapter we have made use of the spotted tiling representation of an n-color composition to show bijections between such compositions and other objects counted by the bisection of the Fibonacci sequence. The bijection to binary words with $v-1$ strictly increasing runs seems to be particularly interesting.

This bijection offers a generalization of the classical bijection between compositions of n and binary words of length $n-1$. This classical bijection has been extremely useful in many bijective arguments between sets of different types of compositions.

Much information regarding the n-color composition can be readily obtained from the corresponding binary word. For instance, the number of parts of the composition is one plus the number of runs $|01|$. One can compare this observation with the classical case (1.1) where the number of parts is one plus the number of 0's.

Also, by taking the "conjugate" of a binary word (i.e. exchange 1 and 0), the classical result yields a number of interesting one-to-one correspondences between compositions with various constraints. We note here that by taking the "conjugate" of a binary word, the number of strictly increasing runs will change by at most 1. This number will stay the same if the original word starts and ends with the same digit. This gives us a potential candidate for the definition of the conjugate of an

n-color composition, although this conjugate may differ by 1 from the original n-color composition.

CHAPTER 3

SOME ANALYTIC RESULTS

Here we explore some results with the use of generating functions. Generating functions are useful because they allow us to examine a sequence analytically, using the highly-studied properties of formal power series. We can write an infinite series as a compact formula, and use familiar properties to find new results.

3.1 Generating Functions

Definition 3.1.1. *The generating function of a sequence* $\{a_0, a_1, a_2, \ldots\}$*, which we will denote with a script, is defined as*

$$\begin{aligned}\mathcal{A}(x) &= a_0 \cdot x^0 + a_1 \cdot x^1 + a_2 \cdot x^2 + \cdots \\ &= \sum_{n=0}^{\infty} a_n \cdot x^n.\end{aligned}$$

We illustrate this with two examples we will use in Section 3.2.

Example 3.1.2. *The generating function of the simple sequence* $\{0, 1, 1, 1, \ldots\}$ *is*

$$\begin{aligned}\mathcal{A}(x) &= 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \cdots \\ &= x + x^2 + x^3 + \cdots \\ &= \frac{x}{1-x}.\end{aligned}$$

Example 3.1.3. *The generating function of the sequence* $\{0, 1, 2, 3, \ldots\}$ *is*

$$\begin{aligned}\mathcal{B}(x) &= 0 \cdot x^0 + 1 \cdot x^1 + 2 \cdot x^2 + 3 \cdot x^3 + \cdots \\ &= x + 2 \cdot x^2 + 3 \cdot x^3 + \cdots \\ &= x \cdot \left(\frac{d}{dx}(1 + x + x^2 + x^3 + \cdots)\right) \\ &= x \cdot \left(\frac{d}{dx}\frac{1}{1-x}\right) \\ &= \frac{x}{(1-x)^2}.\end{aligned}$$

In order to show the power of generating functions, we will now find the number of compositions of n using 3 parts and then generalize the result to k parts and prove Theorem 1.1.1 another way.

Consider the formal power series

$$(x^1 + x^2 + x^3 + \cdots)(x^1 + x^2 + x^3 + \cdots)(x^1 + x^2 + x^3 + \cdots). \tag{3.1}$$

If we expand and combine like terms, we obtain

$$\begin{array}{ll} & x^{1+1+1} \\ + & (x^{1+1+2} + x^{1+2+1} + x^{2+1+1}) \\ + & (x^{1+1+3} + x^{1+3+1} + x^{3+1+1} + x^{1+2+2} + x^{2+1+2} + x^{2+2+1}) \\ + & (x^{1+1+4} + x^{1+4+1} + x^{4+1+1} + x^{1+2+3} + x^{1+3+2} \\ & + x^{3+1+2} + x^{2+1+3} + x^{2+3+1} + x^{3+2+1} + x^{2+2+2}) \\ + & \cdots . \end{array}$$

Examining the exponents of x, we see that written this way, they are exactly the compositions with 3 parts. Denoting $\mathcal{C}_k(x)$ as the generating function for

$$\{C_k(1),\ C_k(2),\ C_k(3),\ \ldots\}$$

we see from (3.1) and Theorem 3.1.2 that

$$\begin{aligned} \mathcal{C}_3(x) &= C_3(1)x^1 + C_3(2)x^2 + C_3(3)x^3 + C_3(4)x^4 + \cdots \\ &= \left(\frac{x}{1-x}\right)^3 . \end{aligned}$$

Notice that the smallest exponent of x in $\mathcal{C}_3(x)$ is 3. This is because we cannot write 1 or 2 as a sum of 3 positive integers. Following the same reasoning, we can obtain the generating function for the number of compositions of n using k parts:

$$\begin{aligned}\mathcal{C}_k(x) &= \overbrace{(x^1+x^2+x^3+\cdots)(x^1+x^2+x^3+\cdots)\cdots(x^1+x^2+x^3+\cdots)}^{k\ terms}\\ &= \left(\frac{x}{1-x}\right)^k.\end{aligned}$$

Now we are ready to obtain $\mathcal{C}(x)$, the generating function of

$$\{C(1),\ C(2),\ C(3),\ \ldots\}$$

and use this to get a formula for $C(n)$ which gives another proof of Theorem 1.1.1. Since a composition can have either 1, 2, 3, ... parts, we can count the total number of compositions of n by summing the number of compositions with 1, 2, 3, ... parts. Doing this yields

$$\begin{aligned}\mathcal{C}(x) &= \sum_{n=0}^{\infty} C(n)x^n\\ &= (x^1+x^2+x^3+\cdots)\\ &\quad +(x^1+x^2+x^3+\cdots)^2\\ &\quad +(x^1+x^2+x^3+\cdots)^3\\ &\quad +\cdots\\ &= \left(\frac{x}{1-x}\right)+\left(\frac{x}{1-x}\right)^2+\left(\frac{x}{1-x}\right)^3+\cdots\\ &= \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}\\ &= \frac{x}{1-2x}\\ &= x+2x^2+4x^3+8x^4+\cdots\\ &= \sum_{n=1}^{\infty} 2^{n-1}x^n.\end{aligned}$$

Equating the coefficients of x^n we see that $C(0)=0$ and for $n \geq 1$, $C(n) = 2^{n-1}$.

3.2 Some known results

Using the technique just demonstrated, we can show a useful result from [5] for determining the number of compositions of an integer from a more generalized set than the natural numbers. For instance, the set of naturals in which each integer of size k has k different colors to choose from.

Suppose in a more general sense that each integer n has $f(n)$ colors to choose from, where f is some function from $\mathbb{N}$ to $\mathbb{Z}_{\geq 0}$. Let $I_f(x)$ be the generating function for the sequence $f(1),\ f(2),\ f(3),\ \ldots$.

Theorem 3.2.1. *The generating function for the number of $f(n)$-color compositions of v is given by*

$$\mathcal{C}(x) = \frac{I_f(x)}{1 - I_f(x)}.$$

Proof. Using the technique demonstrated in Section 3.1, we have

$$\begin{aligned}
\mathcal{C}(x) &= (f(1)x + f(2)x^2 + \cdots) + (f(1)x + f(2)x^2 + \cdots)^2 + \cdots \\
&= I_f(x) + [I_f(x)]^2 + [I_f(x)]^3 + \cdots \\
&= \frac{I_f(x)}{1 - I_f(x)}.
\end{aligned}$$

□

We will use this theorem to find the generating functions of several more types of compositions. We can quickly find the generating function for the number of compositions and n-color compositions by using Examples 3.1.2, 3.1.3 and Theorem 3.2.1 as below. Here we let $\mathcal{C}(x)$ denote the generating function for the number of compositions of n and $\mathcal{CC}(x)$ denote the corresponding generating function for n-color compositions. These were first proven in [1].

Theorem 3.2.2.

$$\begin{aligned}\mathcal{C}(x) &= \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} \\ &= \frac{x}{1-2x}\end{aligned}$$

Theorem 3.2.3.

$$\begin{aligned}\mathcal{CC}(x) &= \frac{\frac{x}{(1-x)^2}}{1-\frac{x}{(1-x)^2}} \\ &= \frac{x}{(1-x)^2-x} \\ &= \frac{x}{1-3x+x^2}.\end{aligned}$$

and the corresponding recurrence relation is given by

$$CC(n) = 3 \cdot CC(n-1) - CC(n-2).$$

The generating function $\mathcal{CC}(x)$ is also the generating function for the sequence for the bisection of the Fibonacci numbers. Hence

$$CC(v) = F_{2v}.$$

A nice bijection was shown in [9] between the n-color compositions of v and the ordinary compositions of $2v$ using only odd parts.

Below, we will provide the generating function for a few types of restricted n-color compositions. These have been used to find nice relationships with other well-known sequences in [1, 9].

Theorem 3.2.4. *The generating function for the number of n-color compositions with only odd parts is given by*

$$\frac{x+x^3}{x^4-x^3-2x^2-x+1}.$$

Proof. The generating function for the set of odd n-color integers is given by

$$\begin{aligned}
&1\cdot x+3\cdot x^3+5\cdot x^5+\cdots\\
=&x\cdot\frac{d}{dx}\frac{x}{1-x^2}\\
=&x\cdot\left(\frac{1-x^2+2x^2}{(1-x^2)^2}\right)\\
=&\frac{x+x^3}{(1-x^2)^2}
\end{aligned}$$

Now using Theorem 3.2.1, the generating function for the number of n-color compositions of v is thus given by

$$\begin{aligned}
&\frac{\frac{x+x^3}{(1-x^2)^2}}{1-\frac{x+x^3}{(1-x^2)^2}}\\
=&\frac{x+x^3}{(1-x^2)^2-(x+x^3)}\\
=&\frac{x+x^3}{x^4-x^3-2x^2-x+1}
\end{aligned}$$

□

Remark 3.2.5. *The number of n-color compositions of v using only even parts is denoted by $CCo(v)$. The associated recurrence relation for this generating function,*

$$CCo(v)=CCo(v-1)+2\cdot CCo(v-2)+CCo(v-3)-CCo(v-4)$$

with $CCo(0)=0$, $CCo(1)=1$, $CCo(2)=1$, $CCo(3)=4$, was proved bijectively in [9].

Theorem 3.2.6. *The generating function for the number of n-color compositions using only even parts is*

$$\frac{2x^2}{x^4-4x^2+1}.$$

Proof. We start as in previous proofs by examining the generating function for the sequence of n-color integers we are using.

$$\begin{aligned}
&2\cdot x^2+4\cdot x^4+6\cdot x^6+\cdots\\
=&x\cdot\frac{d}{dx}\frac{x^2}{1-x^2}\\
=&x\cdot\left(\frac{2x(1-x^2)+2x^3}{(1-x^2)^2}\right)\\
=&\frac{2x^2}{(1-x^2)^2}
\end{aligned}$$

And from Theorem 3.2.1, we can obtain the generating function we are looking for:

$$\begin{aligned}
&\frac{\frac{2x^2}{(1-x^2)^2}}{1-\frac{2x^2}{(1-x^2)^2}}\\
=&\frac{2x^2}{(1-x^2)^2-2x^2}\\
=&\frac{2x^2}{x^4-4x^2+1}
\end{aligned}$$

□

Remark 3.2.7. *The number of n-color compositions of v using only even parts is denoted by $CCe(v)$. The corresponding recurrence relation is*

$$CCe(n)=4\cdot CCe(n-2)-CCe(n-4)$$

with $CCe(0)=0$ and $CCe(2)=2$ which was proved in [9]. Note that $CCe(v)=0$ if v is odd.

Theorem 3.2.1 is a powerful tool for quickly determining the generating function for the number of many generalized and/or restricted compositions. We provide one more example below.

Theorem 3.2.8. *The number of n-color compositions of v with parts congruent to r modulo k has the generating function*

$$\frac{rx^r+(k-r)x^{k-r}}{(1-x^k)^2-(rx^r+(k-r)x^{k-r})}.$$

Proof. The set of admissible integers has generating function

$$\begin{aligned}
&rx^r + (k+r)x^{k+r} + (2k+r)x^{2k+r} + \cdots \\
=&x\frac{d}{dx}\left(\frac{x^r}{1-x^k}\right) \\
=&x\frac{rx^{r-1}(1-x^k) + kx^{r+k-1}}{(1-x^k)^2} \\
=&\frac{rx^r - rx^{r+k} + kx^{r+k}}{(1-x^k)^2} \\
=&\frac{rx^r + (k-r)x^{r+k}}{(1-x^k)^2}
\end{aligned}$$

and thus the desired generating function is

$$\begin{aligned}
&\frac{\frac{rx^r+(k-r)x^{r+k}}{(1-x^k)^2}}{1-\frac{rx^r+(k-r)x^{r+k}}{(1-x^k)^2}} \\
=&\frac{rx^r + (k-r)x^{r+k}}{(1-x^k)^2 - (rx^r + (k-r)x^{r+k})}
\end{aligned}$$

□

Remark 3.2.9. *If we set $r = 1$ and $k = 2$ we get exactly Theorem 3.2.4. Similarly, setting $r = 0$ and $k = 2$ we get Theorem 3.2.6.*

3.3 Some observations

Another topic one might wish to study would be the palindromic n-color compositions. Suppose that we are only concerned with the composition being palindromic with respect to the part sizes, but not necessarily colors. Then we can immediately see that the number of n-color compositions of $2v$ with an even number of parts must be the number of n^2-color compositions of v. Using this concept as a motivation, we will explore some general colored compositions in the following chapter. Of course one could also study such palindromic compositions more in depth by considering the number of all such compositions (not just with an even number of parts) by examining the size of the middle part and summing up all the possibilities.

CHAPTER 4

MORE GENERALIZED COLOR COMPOSITIONS

In this chapter we consider some further generalizations of colored compositions. We demonstrate some more bijections, including one which generalizes to a class of colored compositions. Namely, $\binom{n+k-1}{k}$-color compositions of v and $0, 1, 2, 3, \ldots, k+1$ strings of length $v-1$ which avoid decreasing 2-strings except those starting with $k+1$.

4.1 Bijection between $\binom{n}{2}$-color compositions and 01- and 12-avoiding ternary strings

Here we show a bijection between $\binom{n}{2}$-color compositions of v and $0, 1, 2$ strings of length $v-2$ without sequential digits (i.e. 01 and 12 are disallowed). $\binom{n}{2}$-color compositions are similar to n-color compositions, except that now instead of choosing among n colors for a part of size n, we now choose among $\binom{n}{2}$ colors for each part of size n. We note for completeness that the generating function of $\binom{n}{2}$-color compositions is given by (A095263 in [13])

$$\frac{x^2}{-x^3+2x^2-3x+1}.$$

We first need to generalize the spotted tiling we used previously to now include two spots on each tile, with no more than one spot per 1×1 square. Note that since we need to place two dots on each tile, there cannot be any parts of size 1 (This makes sense because $\binom{1}{2} = 0$). Instead of denoting these colorings as n_k with $1 \leq k \leq \binom{n}{2}$, we label them as $n_{i,j}$ with $1 \leq i < j \leq n$. Note that these are equivalent, we simply use the latter to make the transition to a tiling less complicated. The corresponding spotted tile for $n_{i,j}$ is a $1 \times n$ tile with spots in the i^{th} and j^{th} squares. For instance the $\binom{n}{2}$-color composition

$$2_{1,2} + 3_{2,3} + 4_{1,3} + 3_{1,3} = 12$$

is represented by Figure 4.1.

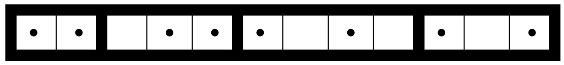


Figure 4.1: Tiling representation of $2_{1,2} + 3_{2,3} + 4_{1,3} + 3_{1,3} = 12$.

We will now describe the bijection from the set of $\binom{n}{2}$-color compositions of length v to the set of $0, 1, 2$ strings of length $v-2$ which avoids 01 and 12 by considering the tiling representation of the compositions. The bijection is described algorithmically in the sense that at each step, we produce an element for the string, and are left with a $\binom{n}{2}$-color composition which is smaller than the previous step. See Figure 4.2 for illustrations of the rules below.

- If the left most part is $2_{1,2}$, we remove the part and generate 02 for the string.
- If the left most square does not contain a dot, we remove it and generate 2 for the string.
- If the left most square does contain a dot, but the square directly to its right does not, we remove the square and move the dot to the square to its right and generate 1 for the string.
- If the left most part is $k_{1,2}$ for $k > 2$, we move both dots one square to the right, remove the leftmost square, and generate 0 for the string.

At the end of the process we will be left with a 1×2 tiling, which we simply remove and do not generate anything.

This figure also shows why we must avoid the 01 and 12 pairs. After generating a 0, we are left with a composition in which the two left-most squares each contain a dot. Thus the next number must be another 0 (or 02). Similarly, after generating a

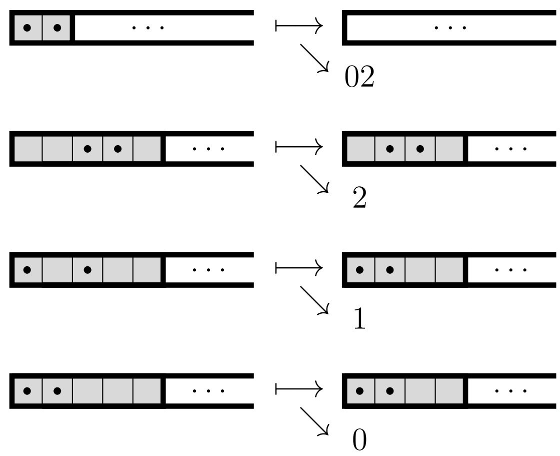


Figure 4.2: Generate a ternary word from an $\binom{n}{2}$-color composition.

1 for the string, we must have a dot in the left-most square. Hence the next number, must be a 0 or 1 (or 02).

Example 4.1.1. *The ternary string corresponding to the* $\binom{n}{2}$*-color composition*

$$2_{1,2} + 3_{2,3} + 4_{1,3} + 3_{1,3}$$

is

$$|02|2|02|1|0|02|1|.$$

4.2 Bijection between $\binom{n+1}{2}$-color compositions of v and $0, 1, 2, 3$-strings of length $v - 1$

In this section we will use a similar notation to the $\binom{n}{2}$-color compositions. However this time, the subscripts (which represent colors) have a slightly different restriction. We represent the $\binom{n+1}{2}$ colors as i, j with $1 \leq i \leq j \leq n$. Allowing $i = j$ is the only difference. This is modeled with the spotted tiling representation by allowing both spots in a part to occupy the same square. Then for example, the $\binom{2+1}{2}$-color composition

$$2_{1,1} + 3_{2,3} + 4_{2,2} + 3_{3,3} = 12$$

is represented by Figure 4.3.

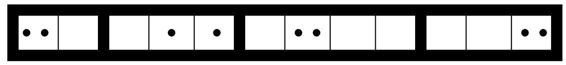


Figure 4.3: Tiling representation of $2_{1,1} + 3_{2,3} + 4_{2,2} + 3_{3,3} = 12$.

The bijection we found is a map from the set of such tilings of size v to the set of $0, 1, 2, 3$-strings of length $v - 1$ which avoid decreasing pairs, unless the pair starts with a 3 (i.e. 10, 20, and 21 are not allowed). To map a given tiling to a string, we examine the internal vertical lines in the tiling representation, and map each one to a 0, 1, 2, or 3 according to the two simple rules below. We will describe the inverse in more detail in Section 4.3.

- If the vertical line is bold (represents the end of a part), we generate a 3 for the string.
- If the vertical line is not bold, we count the number of dots between the line and the next bold line to the left, and generate that number for the string.

We illustrate this in Figure 4.4 below.

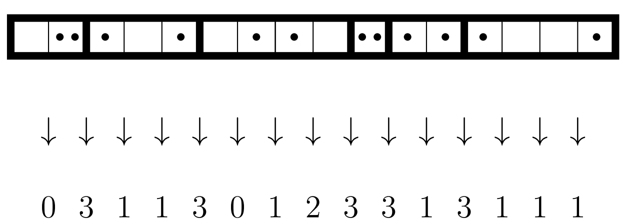


Figure 4.4: Mapping $2_{2,2} + 3_{1,3} + 4_{2,3} + 1_{1,1} + 2_{1,2} + 4_{1,4} = 16$ to 031130123313111.

Remark 4.2.1. *This figure shows why* 10, 20, *and* 21 *are avoided. Once we pass* 1 *dot in a given tile, we cannot pass* 0 *again until we start at the next tile. The same reasoning explains why* 20 *and* 21 *are also avoided.*

4.3 A generalized bijection

After finding the simple rules to describe the $\binom{n+1}{2}$-color composition, a natural question is whether such rules generalize. If we consider allowing k dots per tile, we can find a generalization for the bijection described in Section 4.2. Since there are $\binom{n+k-1}{k}$ ways to arrange k dots in n boxes, this is naturally the next set of colored compositions we consider. We label each part of size n with a subscript as done previously, $n_{j_1,j_2,\ldots,j_{k-1},j_k}$ with $1 \leq j_1 \leq j_2 \leq \ldots \leq j_{k-1} \leq j_k \leq n$ and the j_i correspond to the locations of the dots in our tiling.

Below, we describe the map from the set of all $\binom{n+k-1}{k}$-color compositions of v (for a given k) to the set of all strings of length $v-1$ with characters chosen from the set $\{0, 1, 2, \ldots, v, v+1\}$ which avoid all decreasing pairs, except when $v+1$ is included (any number from the set can go after $v+1$, but only larger numbers can go after any other number).

Again, we examine the internal vertical lines of the tiling.

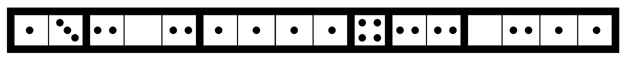
- If the vertical line is bold (represents the end of a part), we generate a $k+1$ for the string.
- If the vertical line is not bold, we count the number of dots between the line and the next bold line to the left, and generate that number for the string.

We can find the inverse of an admissible string with a given k by the following procedure.

1. First, count the number of digits, say m and then draw a tiling with $m+1$ boxes where each internal vertical line corresponds to a digit.
2. Locate the $k+1$ digits in the string and make the vertical lines corresponding to them bold to represent the end of each tile.

3. Place k spots in each tile such that the leftmost non-$k+1$ digit, say j has j spots to it's left in the tile, and continue the process until reaching the end of each tile. If fewer than k spots are used in any tile, place enough spots in the right-most square in the tile to reach k spots.

$$2_{1,2,2,2} + 3_{1,1,3,3} + 4_{1,2,3,4} + 1_{1,1,1,1} + 2_{1,1,2,2} + 4_{2,2,3,4} = 16$$



$$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$$

$$1\ 5\ 2\ 2\ 5\ 1\ 2\ 3\ 5\ 5\ 2\ 5\ 0\ 2\ 3$$

Figure 4.5: Mapping $2_{1,2,2,2} + 3_{1,1,3,3} + 4_{1,2,3,4} + 1_{1,1,1,1} + 2_{1,1,2,2} + 4_{2,2,3,4} = 16$ to 152251235525023.

If we consider this bijection with $k = 0$, then we are looking at $\binom{n-1}{0}$-color compositions of v. Since $\binom{n-1}{0} = 1$ this is just ordinary compositions, which have a classic tiling representation. Our bijection then maps to binary strings of length $v-1$ which avoid decreasing pairs, except those starting with 1 (i.e. no pairs are avoided). We simply switch the 0's and 1's and get the same classic bijection from Section 1.1.

Similarly, if $k = 1$, we have $\binom{n}{1} = n$-color compositions. After some examination, the bijection we describe in Section 2.3 is almost the same. If we switch the 0's and 2's with each other, we get the exact same thing. Instead of 12-avoiding strings, we get 10-avoiding strings, which is what the general bijection describes.

CHAPTER 5

CONCLUDING REMARKS

The analytic techniques demonstrated in Chapter 3 can be used for many further investigations. Using Theorem 3.2.1, one can find the generating function for many different types of restricted colored compositions. Some other types of compositions one might wish to explore are palindromic n-color compositions, n-color compositions with certain restrictions of parts. We believe there are many more interesting identities and properties yet to be discovered, and looking to the results of ordinary compositions will give insights into what types of interesting identities may yet be discovered. Analytic techniques may then be used to prove many interesting results, and once proven, the spotted tiling representation may be used to find bijective proofs of more identities, giving even more insight into the fascinating properties of n-color compositions.

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