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Quantum States Localized on Lagrangian Submanifolds

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Quantum states

(L, ϖ): Kostant-Souriau line bundle over symplectic manifold (X, ω).

Definition (Souriau 1990)

A ***quantum state*** is a state m of $\mathrm{Aut}(\mathrm{L})$

State of a group G: function $m : \mathrm{G} \to \mathbf{C}$ such that ❶ $m(e) = 1$, ❷ the sesquilinear form

$$(c, d)_m := \sum_{g,h \in \mathrm{G}} \overline{c}_g d_h m(g^{-1}h)$$

on $\mathbf{C}[\mathrm{G}] = \{\text{functions } \mathrm{G} \to \mathbf{C} \text{ with finite support}\}$, is positive.

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$$(c, d)_m := \sum_{g,h \in G} \overline{c}_g d_h m(g^{-1}h) \gg 0.$$

Gives rise to unitary G-module $\mathrm{GNS}_m \ni \varphi$ such that $m(g) = (\varphi, g\varphi)$. (Put $(\cdot, \cdot)_m$ on $\mathbf{C}[G]$, divide out null vectors and complete; $\varphi = [\delta^e]$.)

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A ***quantum state*** (of Aut(L), for X) is a state m of Aut(L) such that

$$\Big|\sum_{j=1}^{n} c_j\, m(\exp(\mathrm{Z}_j))\Big| \leqslant \sup_{x\in\mathrm{X}}\Big|\sum_{j=1}^{n} c_j\, \mathrm{e}^{\mathrm{i}\mathrm{H}_j(x)}\Big|$$

for all choices of $n \in \mathbf{N}$, $c_j \in \mathbf{C}$ and *complete*, *commuting* $\mathrm{Z}_j \in \mathrm{aut}(\mathrm{L})$ with hamiltonians H_j: $\quad \mathrm{H}_j(x) = \varpi(\mathrm{Z}_j(\xi))$.

- A ***quantum representation*** (of Aut(L), for X) is a unitary Aut(L)-module $\mathscr{H}$ s.t. $m(g) = (\varphi, g\varphi)$ is quantum $\forall$ unit $\varphi \in \mathscr{H}$.
- **Theorem** (Souriau). m quantum $\Rightarrow$ GNS_m quantum.

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Examples

None. (Unless X is zero-dimensional.)

Remark. X is a coadjoint orbit of Aut(L). We might more modestly ask for states and representations of smaller groups (of which X is a coadjoint orbit).

Quantum states

X: coadjoint orbit of a connected Lie group G.

Definition (Souriau 1990)

A ***quantum state*** (of G, for X) is a state m of G such that

$$\left|\sum_{j=1}^{n} c_j\, m(\exp(\mathrm{Z}_j))\right| \leqslant \sup_{x\in\mathrm{X}}\left|\sum_{j=1}^{n} c_j\, \mathrm{e}^{\mathrm{i}\langle x,\mathrm{Z}_j\rangle}\right|$$

for all choices of $n \in \mathbf{N}$, $c_j \in \mathbf{C}$ and *commuting* $\mathrm{Z}_j \in \mathfrak{g}$.

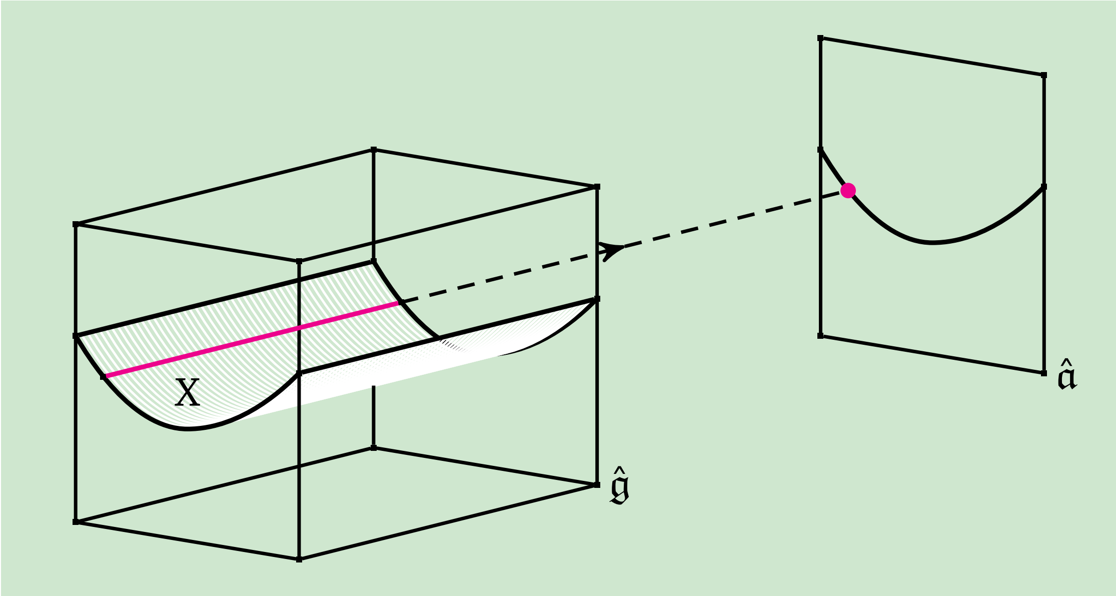
Examples

Too many. (Unless X is zero-dimensional.)

- If $\mathrm{X} = \{x\}$ is an integral point-orbit, then the unique quantum state for X is the character $m(\exp(\mathrm{Z})) = \mathrm{e}^{\mathrm{i}\langle x,\mathrm{Z}\rangle}$.

The statistical interpretation

Let $\hat{\mathfrak{g}}$:= (compact) character group of the *discrete* additive group $\mathfrak{g}$.
We have a dense inclusion $\mathfrak{g}^* \hookrightarrow \hat{\mathfrak{g}}$, $x \mapsto e^{i\langle x,\cdot\rangle}$, and projections



Theorem

A state m of G is quantum for X $\Leftrightarrow$ for each abelian $\mathfrak{a} \subset \mathfrak{g}$, the state $m \circ \exp_{|\mathfrak{a}}$ of $\mathfrak{a}$ has its spectral measure concentrated on $\mathrm{bX}_{|\mathfrak{a}}$, the projection (in $\hat{\mathfrak{a}}$) of the closure bX of X (in $\hat{\mathfrak{g}}$).

This ***spectral measure*** is the probability measure μ on $\hat{\mathfrak{a}}$ such that $(m \circ \exp_{|\mathfrak{a}})(Z) = \int_{\hat{\mathfrak{a}}} \chi(Z)\, d\mu(\chi)$. (Bochner.)

Why "too many" quantum representations?

Because this ('Bohr') closure operation b is *drastic*:

Theorem (Howe-Z., dx.doi.org/10.1017/etds.2013.73)

(a) *If* G *is noncompact simple, every nonzero coadjoint orbit is Bohr dense in* $\hat{\mathfrak{g}}$, *i.e.* $\mathrm{b}X = \hat{\mathfrak{g}}$.

(b) *If* G *is connected nilpotent, every coadjoint orbit is Bohr dense in its affine hull.*

Corollary

(a) *If* G *is noncompact simple,* ***every*** *unitary representation of* G *is quantum for* ***every*** *nonzero coadjoint orbit* (!)

(b) *If* G *is connected nilpotent and* X *spans* $\mathfrak{g}^*$ *(reduce to this case by dividing out* $\mathrm{ann}(X)$*), a unitary representation of* G *is quantum for* X $\Leftrightarrow$ *the center acts in it by the character* $\exp(Z) \mapsto e^{i\langle X,Z\rangle}$.

Localized states

So Souriau's definition is not restrictive enough. 3 ways to proceed:

1. Hope that the much-needed selection will arise by restricting attention to states that *extend* to the whole Aut(L).
2. *Suppress* the Bohr closure implicit in the definition. For results along this line see arxiv.org/abs/1011.5056.
3. Take this closure seriously, because it allows *interesting states*:

Definition

Let $H \subset G$ be a closed subgroup and $Y \subset X_{|\mathfrak{h}}$ a coadjoint orbit of H. A quantum state m for X is ***localized at*** $Y \subset \mathfrak{h}^*$ if the restriction $m_{|H}$ is a quantum state for Y.

We also say that the state is ***localized on*** $\pi^{-1}(Y)$, where π is the projection $X \to \mathfrak{h}^*$. One knows this set is generically a *coisotropic submanifold* — hence at least half-dimensional, and suitable for localizing a system on. We'll mostly consider $Y = \{pt\}$.

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Definition

Let $\mathrm{H} \subset \mathrm{G}$ be a closed subgroup and $\mathrm{Y} \subset \mathrm{X}_{|\mathfrak{h}}$ a coadjoint orbit of H. A quantum state m for X is ***localized at*** $\mathrm{Y} \subset \mathfrak{h}^*$ if the restriction $m_{|\mathrm{H}}$ is a quantum state for Y.

One should expect uniqueness of such a state when $\pi^{-1}(\mathrm{Y})$ is *lagrangian* (half-dimensional): Weinstein (1982) called attaching state vectors to lagrangian submanifolds the FUNDAMENTAL QUANTIZATION PROBLEM.

Nilpotent groups

G : connected, simply connected nilpotent Lie group,
X : coadjoint orbit of G,
x : chosen point in X.
A connected subgroup $\mathrm{H} \subset \mathrm{G}$ is ***subordinate to*** x if, equivalently,

- $\{x_{|\mathfrak{h}}\}$ is a point-orbit of H in $\mathfrak{h}^*$
- $\langle x, [\mathfrak{h}, \mathfrak{h}]\rangle = 0$
- $\mathrm{e}^{ix \circ \log}{}_{|\mathrm{H}}$ is a character of H.

Theorem

Let $\mathrm{H} \subset \mathrm{G}$ *be maximal subordinate to* $x \in \mathrm{X}$*. Then there is a unique quantum state for* X *localized at* $\{x_{|\mathfrak{h}}\} \subset \mathfrak{h}^*$*, namely*

$$m(g) = \begin{cases} \mathrm{e}^{ix \circ \log}(g) & \textit{if } g \in \mathrm{H}, \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover $\mathrm{GNS}_m = \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} \, \mathrm{e}^{ix \circ \log}{}_{|\mathrm{H}}$ (*discrete induction*).

$\mathfrak{a} \subset \mathfrak{h} \;\Rightarrow\; x_{|\mathfrak{a}}$ certain; $\qquad \mathfrak{a} \pitchfork \mathfrak{h} \;\Rightarrow\; x_{|\mathfrak{a}}$ equidistributed in $\hat{\mathfrak{a}}$.

Nilpotent groups

Remark

Kirillov (1962) used $\mathrm{I}(x, \mathrm{H}) := \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \mathrm{e}^{ix \circ \log}{}_{|\mathrm{H}}$ (usual induction). This is

(a) irreducible $\Leftrightarrow$ H is a ***polarization at*** x (: subordinate subgroup such that the bound $\dim(\mathrm{G}/\mathrm{H}) \geqslant \frac{1}{2}\dim(\mathrm{X})$ is attained);

(b) ***equivalent*** to $\mathrm{I}(x, \mathrm{H}')$ if $\mathrm{H} \neq \mathrm{H}'$ are two polarizations at x.

In contrast:

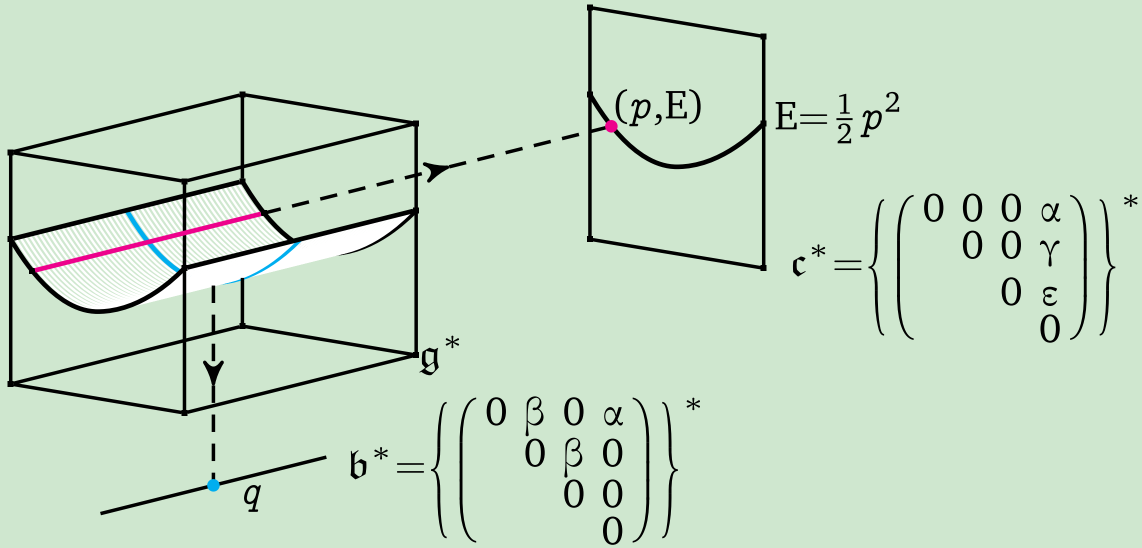
Theorem

Let $\mathrm{H} \subset \mathrm{G}$ *be subordinate to* x. *Then* $\mathrm{i}(x, \mathrm{H}) := \mathrm{ind}_{\mathrm{H}}^{\mathrm{G}} \mathrm{e}^{ix \circ \log}{}_{|\mathrm{H}}$ *is*

(a) *irreducible* $\Leftrightarrow$ H *is* ***maximal subordinate*** *to* x;

(b) ***inequivalent*** *to* $\mathrm{i}(x, \mathrm{H}')$ *if* $\mathrm{H} \neq \mathrm{H}'$ *are two polarizations at* x.

Example: Extended Galilei group $G = \left\{ g = \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & a \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} \right\}$



B and C are maximal subordinate but only C is a polarization.
So $i(x, C)$, $I(x, C)$, $i(x, B)$ are irreducible but $I(x, B)$ is not.

All act by $(g\psi)\binom{r}{t} = e^{-ia} e^{-i\{b(r-c)-\frac{1}{2}b^2(t-e)\}} \psi\binom{r-c-b(t-e)}{t-e}$, but

1. $I(x, B)$ in L^2 functions of $\binom{r}{t}$
2. $I(x, C)$ in L^2 *solutions of Schrödinger's equation* $i\partial_t\psi = \frac{1}{2}\partial_r^2\psi$
3. $i(x, C)$ in *almost periodic solutions*, norm^2 $\lim\limits_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} |\psi|^2 dr$
4. $i(x, B)$ in ℓ^2 functions — no Schrödinger equation needed!

Compact groups

Theorem

Every quantum representation of a compact Lie group G *is continuous. The irreducible with highest weight* λ *is quantum for the coadjoint orbit with dominant element* $\mu \Leftrightarrow \lambda \leqslant \mu$.

So even for compact G, Souriau's definition does not recover the usual 'orbit method' (which posits $\lambda = \mu$). In contrast we have, with $T \subset G$ a maximal torus:

Theorem

- *If* μ *is dominant integral, then there is a unique quantum state* m *for* $X = G(\mu)$ *localized at* $\{\mu_{|\mathfrak{t}}\} \subset \mathfrak{t}^*$; GNS_m *is the irreducible representation with highest weight* μ.
- *If* μ *is dominant and not integral, then there is no such state.*

Euclid's group $G = \left\{ g = \begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} : \begin{smallmatrix} A \in \mathbf{SO}(3) \\ \boldsymbol{c} \in \mathbf{R}^3 \end{smallmatrix} \right\}$

Example: TS^2

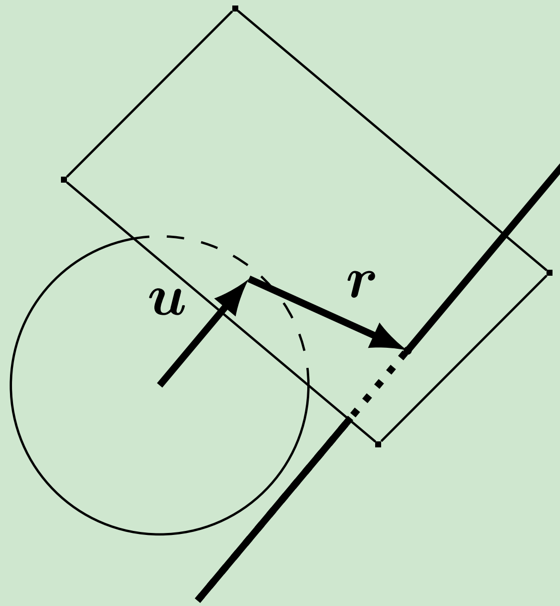
G acts naturally and symplectically on the manifold $X \simeq TS^2$ of oriented lines (a.k.a. light rays) in $\mathbf{R}^3$. 2-form$_{k,s}$:

$$\omega = k\, d\langle \boldsymbol{u}, d\boldsymbol{r} \rangle + s\, \mathrm{Area}_{S^2}.$$

The moment map

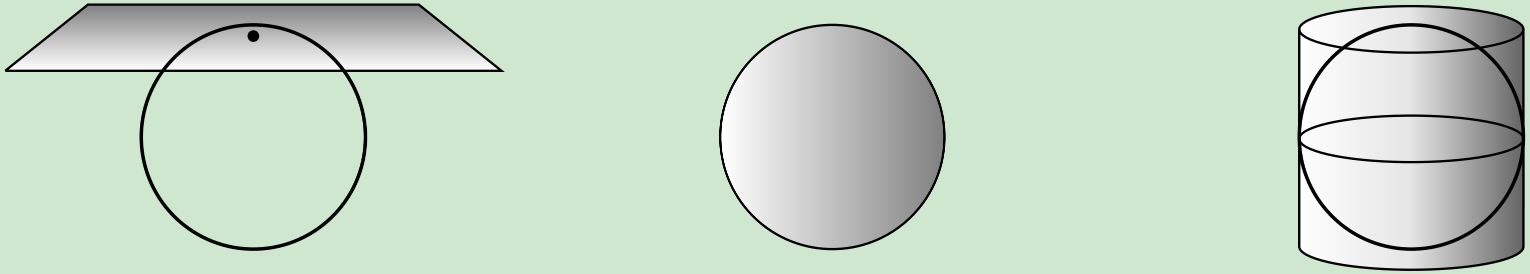
$$\Phi(\boldsymbol{u}, \boldsymbol{r}) = \begin{pmatrix} \boldsymbol{r} \times k\boldsymbol{u} + s\boldsymbol{u} \\ k\boldsymbol{u} \end{pmatrix}$$

makes X into a coadjoint orbit of G.



Case $s = 0$:

We have localized states on 3 types of lagrangians:



(a): the tangent space at the north pole — (b): the zero section — (c): the equator's normal bundle

$$\text{(a)}\quad m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle k\boldsymbol{e}_3, \boldsymbol{c}\rangle} & \text{if } \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(b)}\quad m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \frac{\sin\|k\boldsymbol{c}\|}{\|k\boldsymbol{c}\|}$$

$$\text{(c)}\quad m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(\|k\boldsymbol{c}_\perp\|) & \text{if } \mathrm{A}\boldsymbol{e}_3 = \pm\boldsymbol{e}_3, \\ 0 & \text{otherwise} \end{cases}$$

The resulting GNS modules can be realized as various spaces of solutions of Helmholtz's equation $\Delta\psi + k^2\psi = 0$, with G-action $(g\psi)(\boldsymbol{r}) = \psi(\mathrm{A}^{-1}(\boldsymbol{r} - \boldsymbol{c}))$.

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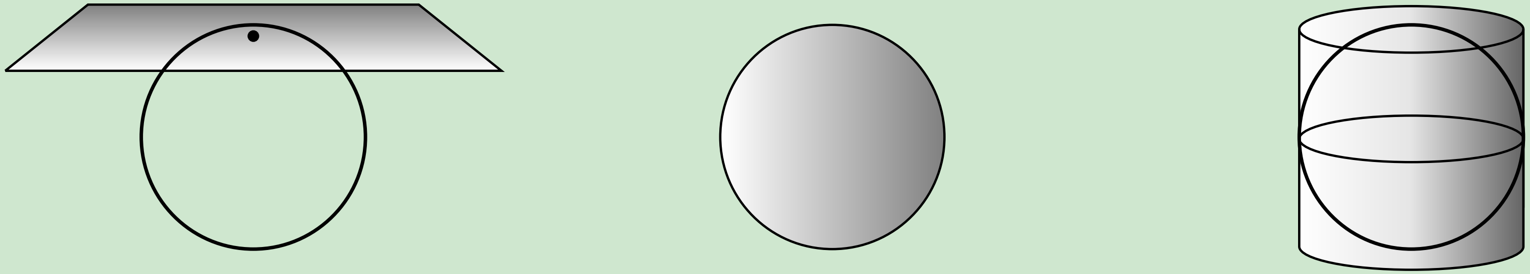

(a): the tangent space at the north pole

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cyclic vector:

(a) $m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle k\boldsymbol{e}_3, \boldsymbol{c}\rangle} & \text{if } A\boldsymbol{e}_3 = \boldsymbol{e}_3, \\ 0 & \text{otherwise.} \end{cases}$ $\quad \psi(\boldsymbol{r}) = e^{-ikz}$

(b) $m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \dfrac{\sin \|k\boldsymbol{c}\|}{\|k\boldsymbol{c}\|}$ $\quad \psi(\boldsymbol{r}) = \dfrac{\sin \|k\boldsymbol{r}\|}{\|k\boldsymbol{r}\|}$

(c) $m\begin{pmatrix} A & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(\|k\boldsymbol{c}_\perp\|) & \text{if } A\boldsymbol{e}_3 = \pm\boldsymbol{e}_3, \\ 0 & \text{otherwise} \end{cases}$ $\quad \psi(\boldsymbol{r}) = J_0(\|k\boldsymbol{r}_\perp\|)$

The resulting GNS modules can be realized as various spaces of solutions of Helmholtz's equation $\Delta\psi + k^2\psi = 0$, with G-action $(g\psi)(\boldsymbol{r}) = \psi(A^{-1}(\boldsymbol{r} - \boldsymbol{c}))$.

Case $s = 1$ (zero section no longer lagrangian):

The unique quantum state localized on the tangent space (a) becomes

$$m\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\alpha} e^{i\langle k\boldsymbol{e}_3, \boldsymbol{c}\rangle} & \text{if } \mathrm{A} = e^{j(\alpha \boldsymbol{e}_3)}, \quad (j(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \times \cdot) \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{GNS}_m = \{\ell^2$ sections $\boldsymbol{b}$ of the tangent bundle $\mathrm{TS}^2 \to \mathrm{S}^2\}$, with G-action $(g\boldsymbol{b})(\boldsymbol{u}) = e^{\langle \boldsymbol{u}, k\boldsymbol{c}\rangle \mathrm{J}} \mathrm{A}\boldsymbol{b}(\mathrm{A}^{-1}\boldsymbol{u})$ where $\mathrm{J}\delta\boldsymbol{u} = j(\boldsymbol{u})\delta\boldsymbol{u}$. Putting

$$\mathbf{F}(\boldsymbol{r}) = (\mathbf{B} + i\mathbf{E})(\boldsymbol{r}) := \sum_{\boldsymbol{u} \in \mathrm{S}^2} e^{-\langle \boldsymbol{u}, k\boldsymbol{r}\rangle \mathrm{J}} (\boldsymbol{b} - i\mathrm{J}\boldsymbol{b})(\boldsymbol{u})$$

one obtains a Hilbert space of almost-periodic solutions of the reduced Maxwell equations

$$\begin{cases} \operatorname{div} \mathbf{B} = 0, & \mathbf{curl}\, \mathbf{B} = k\mathbf{B}, \\ \operatorname{div} \mathbf{E} = 0, & \mathbf{curl}\, \mathbf{E} = k\mathbf{E}, \end{cases}$$

with G-action $(g\mathbf{F})(\boldsymbol{r}) = \mathrm{A}\mathbf{F}(\mathrm{A}^{-1}(\boldsymbol{r} - \boldsymbol{c}))$. The cyclic vector is $\mathbf{F}(\boldsymbol{r}) = e^{-ikz}(\boldsymbol{e}_1 - i\boldsymbol{e}_2)$.