Abstract

A graph $G$ is said to be $H$-saturated if $G$ contains no subgraph isomorphic to $H$ but the addition of any edge between non-adjacent vertices in $G$ creates one. While induced subgraphs are often studied in the extremal case with regard to the removal of edges, we extend saturation to induced subgraphs. We say that $G$ is induced $H$-saturated if $G$ contains no induced subgraph isomorphic to $H$ and the addition of any edge to $G$ results in an induced copy of $H$. We demonstrate constructively that there are non-trivial examples of saturated graphs for all cycles and an infinite family of paths and find a lower bound on the size of some induced path-saturated graphs.

1 Introduction

In this paper we address the problem of graph saturation as it pertains to induced graphs, in particular paths and cycles. We begin with some background and definitions, and complete Section 1 with statements of the main theorems. In Section 2 we demonstrate that there are non-trivial induced saturated graphs for an infinite family of paths, and prove lower bounds on the number of edges in possible constructions. We continue on to demonstrate results regarding induced cycles in Section 3 and claws in Section 4.

A number of results were discovered using the SAGE mathematics software [11], an open source mathematics suite.

Throughout we use $K_n$ to denote the complete graph on $n$ vertices, $C_n$ the cycle on $n$ vertices, $K_{n,m}$ the complete bipartite graph with parts of order $n$ and $m$, and $P_n$ the path on $n$ vertices. All graphs in this paper are simple, and by $\overline{G}$ we mean the complement of the graph $G$. If $u, v$ are non-adjacent vertices in $G$ then $\overline{G} + uv$ is the graph $G$ with edge $uv$ added. Given graphs $G$ and $H$ the graph join $G \vee H$ is composed of a copy of $G$, a copy of $H$, and all possible edges between the vertices of $G$ and the vertices of $H$. The graph union $G \cup H$ consists of disjoint copies of $G$ and $H$. The graph $kH$ consists of the union of $k$ copies of $H$. In particular a matching is a collection of pairwise disjoint edges, denoted $kK_2$. The order $n(G)$ and size $e(G)$ of $G$ are the numbers of its vertices and edges, respectively. A vertex $v$ in a connected graph $G$ is a cut vertex if its removal results in a disconnected graph. If $G$ has no cut vertex then it is 2-connected, and a maximal 2-connected subgraph of $G$ is a block. Note that a cut edge is also a block.

For simple graphs $G$ and $H$ we say that $G$ is $H$-saturated if it contains no subgraph isomorphic to $H$ but the addition of any edge from $\overline{G}$ creates a copy of $H$. We refer to $G$ as the parent graph. The study of graph saturation began when Mantel and students [9] determined the greatest number of edges in a $K_3$-free graph on $n$ vertices in 1907, which was generalized by Turán in the middle of the last century [12] to graphs that avoid arbitrarily large cliques. Erdős, Hajnal, and Moon then addressed the problem of finding the fewest number of edges in a $K_m$-saturated graph [3]. In particular, they proved the following theorem.

**Theorem 1.1.** For $m \geq 3$ and $n \geq m$, the unique smallest graph on $n$ vertices that is $K_m$-saturated is $K_{m-2} \vee \overline{K_{n-m+2}}$. This graph contains $\binom{m-2}{2} + (n - m + 2)(m - 2)$ edges.
Since then, graph saturation has been studied extensively, having been generalized to many other families of graphs, oriented graphs [7], topological minors [5], and numerous other properties. A comprehensive collection of results in graph saturation is available in [4].

Given a graph $G$ and a subset $X$ of vertices of $G$, the subgraph induced by $X$ is the graph composed of the vertices $X$ and all edges in $G$ among those vertices. We say that a subgraph $H$ of $G$ is an induced subgraph if there is a set of vertices in $G$ that induces a graph isomorphic to $H$. We say that $G$ is $H$-free if $G$ contains no induced subgraph isomorphic to $H$.

Finding induced subgraphs of one graph isomorphic to another is a traditionally difficult problem. Chung, Jiang, and West addressed the problem of finding the greatest number of edges in degree-constrained $P_n$-free graphs [1]. Martin and Smith created the parameter of induced saturation number [10]. We include their definition below for completeness.

**Definition 1.2** (Martin, Smith 2012). Let $T$ be a graph with edges colored black, white, and gray. The graph $T$ realizes $H$ if the black edges and some subset of the gray edges of $T$ together include $H$ as an induced subgraph. The induced saturation number of $H$ with respect to an integer $n$ is the fewest number of gray edges in such a graph $T$ on $n$ vertices that does not realize $H$ but if any black or white edge is changed to gray then the resulting graph realizes $H$.

In this paper we only consider adding edges to a simple non-colored graph.

**Definition 1.3.** Given graphs $G$ and $H$ we say that $G$ is induced $H$-saturated if $G$ does not contain an induced subgraph isomorphic to $H$ but the addition of any edge from $G$ to $G$ creates one.

Note that in Definition 1.3 we allow $G$ to be a complete graph. This case provides for a trivial family of induced $H$-saturated graphs for any non-complete graph $H$. Henceforth we will be concerned with determining non-trivial induced $H$-saturated graphs.

### 1.1 Main results

We will prove the following results to show the existence of non-complete induced $P_m$-saturated graphs for infinitely many values of $m$.

**Theorem.** (2.6) For any $k \geq 0$ and $n \geq 14 + 8k$ there is a non-complete induced $P_{9+6k}$-saturated graph on $n$ vertices. Further, if $n$ is a multiple of $(14 + 8k)$ there is such a graph that is 3-regular.

As we will see in Section 2.1, these orders are the result of the search for a longest induced path in a class of vertex transitive hamiltonian graphs of small size, visualizable with high rotational symmetry.

**Theorem.** (2.11) If $G$ is an induced $P_m$-saturated graph on $n$ vertices with no pendant edges except a $K_2$ component, $m > 4$, then $G$ has size at least $\frac{3}{2}(n - 2) + 1$. This bound is realized when $m = 9 + 6k$ and $n \equiv 2 \mod (14 + 8k)$.
Theorem. (2.14) For every integer $k > 0$ there is a non-complete graph that is induced $P_{11+6k}$-saturated.

Regarding cycles, we will prove the following theorem.

Theorem. (3.3) For any $k \geq 3$ and $n \geq 3(k − 2)$ there is a non-complete induced $C_k$-saturated graph of order $n$.

Finally, we will demonstrate the following regarding induced claw-saturated graphs.

Theorem. (4.2) For all $n \geq 12$, there is a graph on $n$ vertices that is induced $K_{1,3}$-saturated and is non-complete.

2 Paths

2.1 An infinite family of paths

The only induced $P_2$-saturated graph on $n \geq 2$ vertices is $K_n$. The induced $P_3$-saturated graph of order $n$ with the smallest size is either the matching $\frac{n}{2}K_2$ if $n$ is even, or $\frac{n−1}{2}K_2 \cup K_1$ if $n$ is odd. The case for $P_4$ is similar, consisting of the matching $\frac{n}{2}K_2$ if $n$ is even and $\frac{n−3}{2}K_2 \cup K_3$ if $n$ is odd. It is also easily seen that the graph of order 9 and size 12 consisting of a triangle with each vertex sharing a vertex with another triangle, as in Figure 1, is induced $P_5$-saturated, and that the Petersen graph is induced $P_6$-saturated.

We begin our analysis of induced path-saturated graphs by examining an infinite family of cubic hamiltonian graphs developed by Lederberg [8] and modified by Coxeter and Frucht, and later by Coxeter, Frucht, and Powers [6, 2]. For our purposes we will only consider graphs from this family denoted in LCF (for Lederberg, Coxeter, Frucht) notation by $[x, −x]^a$ with $x$ odd. A graph of this form consists of a 3-regular cycle on $2a$ vertices $\{v_0v_1...v_{2a−1}\}$ and a matching that pairs each $v_{2i}$ with $v_{2i+x}$, with arithmetic taken modulo $2a$. See Figure 2 for an example.

Let $G_k$ denote the graph with LCF notation $[5, −5]^{7+4k}$. Note that the order of $G_k$ is $14 + 8k$. First we find a long induced cycle in $G_k$.

Fact 2.1. The graph $G_k$ has an induced cycle of length $8 + 6k$. 

Figure 1: An induced $P_5$-saturated graph
Figure 2: The graph $G_2$, which has LCF notation $[5, -5]^{15}$, with an induced $C_{20}$

Proof. Let $n$ be the order of $G$, and let $C = \{v_0, v_1, v_2, v_3, v_{n-2}, v_{n-3}, v_{n-8}, v_{n-9}\}$. Then, proceed to add 6 vertices at a time to $C$ in the following way until $v_5$ is included. Let $v_p$ be the last vertex added to $C$. Add the vertices $\{v_{p-5}, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-7}, v_{p-8}\}$. Once $v_5$ is added, the graph induced by $C$ is a chordless cycle of order $8 + 6k$ (Figure 2).

Note that the closed neighborhoods of $v_{n-5}$ and $v_{n-6}$ are disjoint from the cycle $C$ constructed in the proof of Fact 2.1. Therefore, the addition of any edge between any vertex on this cycle and $v_{n-5}$ or $v_{n-6}$ generates an induced path of order $9 + 6k$. A simple reflection that reverses $v_{n-5}$ and $v_{n-6}$ shows that another induced cycle of the same length exists in $G_k$.

We next must bound the length of induced paths in $G_k$. Note that a simple counting argument is not sufficient, since in general a 3-regular graph on $14 + 8k$ vertices may contain an induced path on as many as $10 + 6k$ vertices as seen in the following construction. Consider a path $P$ on $10 + 6k$ vertices and an independent set $X$ of $4 + 2k$ vertices. From each internal vertex in $P$ add a single edge to a vertex in $X$, and from the endpoints in $P$ add two edges to vertices in $X$, in such a way as to create a 3-regular graph. The resulting graph has order $14 + 8k$ and an induced path on $10 + 6k$ vertices.

Lemma 2.2. The graph $G_k$ contains no induced path on more than $8 + 6k$ vertices.

Proof. First, note that an exhaustive search of $G_9$ yields a longest induced path of order 7. Consider the case where $k \geq 1$. Let $P$ be a longest induced path in $G_k$. Let $V$ be the $m = 8 + 6k$ vertices in the cycle $C$ from the proof of Fact 2.1. Let the sets $U, X, Y$ contain the remaining vertices that have 3, 2, and 0, neighbors, respectively, in $V$. Note that $|X| = 4$ and $|Y| = 2$, irrespective of $k$, and $|U| = 2k$. Consider the induced path $P^0$ on $m$ vertices in Figure 3. For example, in Figure 2 $P^0$ would be the path $v_{26}v_1v_2v_3v_28v_{29}v_{24}v_{23}v_{22}v_{21}v_{20}v_{19}v_{14}v_{13}v_8v_9v_{10}v_{11}v_6v_5$. We claim $P$ is no longer than $P^0$. 
For every vertex \( u \in U \) in \( P \) there is one neighbor (if \( u \) is an internal vertex of \( P \)) or two neighbors (if \( u \) is a terminal vertex) from \( V \) not in \( P \). Let us assume that \( U_P = \{ u_1, \ldots, u_l \} = U \cap P \), and denote by \( N(U_P) \subseteq V \) the neighbors of all vertices in \( U_P \). So \( P \) includes \( l \) vertices from \( U \) and avoids at least \( l \) vertices from \( N(U_P) \).

Similarly, assume there is a vertex \( x \) in \( P \cap V \) that is not in \( P^0 \). Then either a neighbor of \( x \) in \( P^0 \cap V \) is not in \( P \), or a route through \( X \cup Y \) on \( P^0 \) is diverted and hence a vertex from \( P^0 \cap (X \cup Y) \) is not in \( P \). In either case, the inclusion of any vertex from \( U \) in \( P \) leads to at least one fewer vertex from \( P^0 \), and hence does not lead to a longer induced path.

Finally, we consider the possible inclusion of vertices from \( X \cup Y \). It is easily seen that no induced path can include all vertices from \( X \cup Y \). If \( P \) has no vertices from \( X \cup Y \) then it is strictly shorter than \( P^0 \). If \( P \) contains exactly one or two vertices from \( X \) then there are at least three vertices in \( V \) not in \( P \), making \( P \) at least as short as \( P^0 \). Any induced path containing 5 vertices from \( X \cup Y \) must exclude one from \( Y \) as in \( P^0 \), and so no other induced path also containing 5 of these vertices will be longer that \( P^0 \). Therefore, \( P^0 \) is a longest induced path in \( G_k \), and hence \( G_k \) does not contain an induced path on more than \( 8 + 6k \) vertices.

Now we demonstrate that the graph \( G_k \) is saturated with respect to the property of long induced paths.

Lemma 2.3. The graph \( G_k \) is induced \( P_{9+6k} \)-saturated.

Proof. Once again we let \( n = 14 + 8k \), the order of \( G_k \). By Lemma 2.2 there is no induced path of order \( 9 + 6k \) in the graph \( G_k \). Define the bijections \( \phi \) and \( \psi \) on the vertices of \( G_k \) by \( \phi(v_i) = v_{i-1} \) (reflection) and \( \psi(v_i) = v_{i+1} \) (rotation), with arithmetic modulo \( n \). Note that both \( \phi^2 \) and \( \psi \) are automorphisms of \( G_k \). Given \( v_i, v_j \in V(G_k) \) there is an automorphism of the form \( \phi^{2k} \circ \psi^m \) that takes \( v_i \) to \( v_j \), where \( k \) is some integer and \( m \in \{0,1\} \). Therefore, the graph \( G_k \) is vertex transitive under repeated applications of \( \phi \) and \( \psi \). In particular, the
induced cycle $C_k$ can be rotated and reflected via these functions to yield a function $f$ such that for any pair $x, y$ of nonadjacent vertices in $G_k$ there is an image $f(C_k)$ so that $x$ is on $f(C_k)$ and neither $y$ nor its neighbors are on $f(C_k)$. Therefore, between any two nonadjacent vertices the addition of any edge creates an induced $P_{9+6k}$.

We now generalize to an arbitrary number of vertices.

**Lemma 2.4.** The disjoint union $H$ of $m$ copies of $G_k$, with at most one complete graph on at least 2 vertices, is induced $P_{9+6k}$-saturated.

**Proof.** Since each connected component of $H$ is induced $P_{9+6k}$-saturated by Lemma 2.3 we need only consider the addition of edges between components. Since $G_k$ is vertex transitive and contains an induced cycle on $(8+6k)$ vertices (Fact 2.1), each vertex in $G_k$ is the terminal vertex of an induced path on $(7+6k)$ vertices. Therefore, any edge between disjoint copies of $G_k$ creates an induced path on far more than the necessary $(8+6k)$ vertices. The addition of an edge between a copy of $G_k$ and a complete component will also result in an induced $P_{9+6k}$ Therefore, $H$ is induced $P_{9+6k}$-saturated.

**Lemma 2.5.** The join of any complete graph to any induced $P_n$-saturated graph, for any $n \geq 4$, generates a new induced $P_n$-saturated graph.

**Proof.** Note that joining a clique to a graph does not contribute to the length of the longest induced path except in the most trivial cases of $P_1, P_2$, and $P_3$, nor does it add any non-edges to the graph which require testing for saturation.

Note that joining a complete graph to any induced $H$-saturated graph, for any non-complete graph $H$, generates a new induced $H$-saturated graph. Therefore, we can prove the main result of this section.

**Theorem 2.6.** For any $k \geq 0$ and $n \geq 14 + 8k$ there is a non-complete induced $P_{9+6k}$-saturated graph on $n$ vertices. Further, if $n$ is a multiple of $(14 + 8k)$ there is such a graph that is 3-regular.

**Proof.** By Lemma 2.3 there is such a graph on $n = 14 + 8k$ vertices, and Lemmas 2.4 and 2.5 demonstrate that $n$ increases without bound.

### 2.2 Lower bounds

As an analogue to Theorem 1.1 by Erdős, Hajnal, and Moon [3], in which smallest $K_n$-saturated graphs are studied, we now turn our attention to finding the smallest induced $P_m$-saturated graphs. Assume throughout that $m > 3$.

First we look at some properties of induced $P_m$-saturated graphs with pendant edges, and then we will turn our attention to graphs with minimum degree two.

**Fact 2.7.** If $u$ and $v$ are distinct pendant vertices in an induced $P_m$-saturated graph $G$ then the distance from $u$ to $v$ is greater than three.
Proof. If \( u \) and \( v \) share a neighbor \( w \) then the addition of edge \( uv \) cannot create an induced path that includes \( w \), so their distance is at least three. If instead \( u \) has neighbor \( w_u \) and \( v \) has neighbor \( w_v \), with \( w_u \) adjacent to \( w_v \), then the added edge \( uw \) must begin an induced \( P_m \). However, this edge can be replaced in \( G \) by \( vw \), so \( G \) must already contain an induced \( P_m \). Therefore the neighbors of \( u \) and \( v \) cannot be adjacent. \qed

Next we examine the neighbor of a pendant vertex in an induced \( P_m \)-saturated graph.

**Fact 2.8.** Let \( v \) be a pendant vertex in a non-complete component of an induced \( P_m \)-saturated graph \( G \), with neighbor \( u \). Then \( u \) has degree at least four.

Proof. If \( \deg(u) = 2 \) then the addition of the edge joining its neighbors cannot create a longer induced path than one that includes \( u \). Assume \( u \) only has neighbors \( v, a, \) and \( b \). If \( a \) and \( b \) are adjacent then the added edge \( va \) must begin an induced \( P_m \) that avoids \( b \), but we can then replace \( va \) with \( ua \) and get an induced path of the same length. If instead \( a \) and \( b \) are not adjacent then adding edge \( ab \) to \( G \) does not result in an induced path longer than one containing the path \( aub \). Hence \( u \) has at least one other neighbor \( c \). \qed

For the remainder of the section we will consider non-complete graphs without pendant edges.

**Fact 2.9.** If \( G \) is induced \( P_m \)-saturated and contains a vertex \( v \) of degree 2 then the neighbors of \( v \) are adjacent.

Proof. Assume that \( \deg(v) = 2 \) and \( v \) has neighborhood \( \{u, w\} \). If \( u \) is not adjacent to \( w \) then the addition of edge \( uw \) cannot generate a longer induced path than one originally present in \( G \) that includes edges \( uv, vw \). Therefore, \( u \) and \( w \) must already be adjacent. \qed

As noted in the beginning of Section 2.1, a matching with possibly an isolated vertex or a connected component isomorphic to \( K_3 \) constitute an induced \( P_3 \)- and \( P_4 \)-saturated graph, respectively. Note that when \( m > 4 \) an induced \( P_m \)-saturated graph cannot have more than one complete component, as any edge between two such components generates an induced path of order at most 4. We now demonstrate that any induced \( P_m \)-saturated graph on \( n \) vertices, for \( m > 4 \), has average degree at least 3 among its non-complete components.

**Lemma 2.10.** For \( m > 5 \) all non-complete connected components of an induced \( P_m \)-saturated graph with no pendant edges have average degree at least 3.

Proof. Let \( G \) be an induced \( P_m \)-saturated graph. If all vertices of \( G \) have degree 3 or more then the result is clear, so let us assume that \( v \) is a vertex of \( G \) with degree 2 and with neighbors \( u \) and \( w \). By Lemma 2.9 \( u \) and \( w \) are adjacent. Without loss of generality we may assume that \( \deg(w) \leq \deg(u) \). We will consider the cases in which \( \deg(w) = 2 \) and \( \deg(w) > 2 \).

First assume that both \( \deg(u) \) and \( \deg(w) \) are at least 3. We demonstrate that there are sufficiently many vertices of high degree to yield an average degree of at least 3. Let \( a \) be a
neighbor of \( u \) and \( b \) a neighbor of \( w \), with \( a, b \notin \{u, v, w\} \). If \( \deg(w) = 3 \) and \( a \) and \( b \) are distinct then no induced path containing the new edge \( wa \) can be longer than an induced path containing the sub-path \( wua \). If instead \( a = b \) then the addition of \( va \) does not create any induced path not already in \( G \) by means of edge \( wa \). Thus, if \( \deg(u) \geq \deg(w) \geq 3 \) then \( \deg(u) \geq \deg(w) \geq 4 \).

Now we consider the case in which \( \deg(w) = 2 \). Vertex \( u \) is therefore a cut vertex of \( G \).

Note that if there is another block containing \( u \) that is isomorphic to \( K_3 \) then the addition of an edge between two such blocks does not result in a longer induced path than one already present in the graph. If \( \deg(u) = 3 \), with \( u \) adjacent to a vertex \( a \) distinct from \( w \) and \( v \), then adding edge \( va \) to \( G \) does not create any induced path longer than one already present in \( G \) that uses the edge \( ua \). So \( \deg(u) \geq 4 \). Say that \( \{v, w, w', w''\} \) are in the neighborhood of \( u \) and note that, as above, if \( \deg(u') = \deg(w') = 3 \) then the addition of edge \( w'a \) shows that \( G \) is not induced \( P_m \)-saturated. So the graph \( G \) with vertices \( v, w \) removed must also have average degree at least 3, and therefore \( G \) does as well.

Next consider the set \( T \) of vertices of degree 2 whose neighbors all have degree at least 3, and the set \( S \) composed of neighbors of vertices in \( T \). Since vertices in \( S \) all have degree at least 4, if \( t = |T| \leq |S| = s \) then the graph has at least as many vertices with degree greater than three than those with degree 2 and we are done. Assume instead that \( t > s \). Since the two neighbors of each vertex in \( T \) are adjacent, we know that for each vertex in \( T \) there are at least 3 edges in the induced graph \( <T \cup S> \). Hence the average degree among vertices in \( <T \cup S> \) is at least \( \frac{6t}{s+t} > \frac{6t}{2t} = 3 \). Since all other vertices in \( G \) either have degree at least 3 or are part of a distinct triple with total degree at least 9 as shown above, the average degree of any non-complete component of an induced \( P_m \)-saturated graph is at least 3.

This leads us to the proof of the lower bound for the size of a class of induced \( P_m \)-saturated graphs.

**Theorem 2.11.** If \( G \) is an induced \( P_m \)-saturated graph on \( n \) vertices with no pendant edges except for a \( K_2 \) component, \( m > 5 \), then \( G \) has size at least \( \frac{3}{2}(n - 2) + 1 \). This bound is realized when \( m = 9 + 6k \) and \( n \equiv 2 \mod (14 + 8k) \).

**Proof.** In the graph \( G \) all but at most one connected component consists of vertices of average degree at least 3, with potentially one component isomorphic to \( K_2 \) or \( K_3 \) by Lemma 2.10. Therefore, \( e(G) \geq \frac{3}{2}(n - 2) + 1 \). By Lemma 2.4 the graph consisting of disjoint copies of \( G_k \) and a \( K_2 \) has size \( \frac{3}{2}(n - 2) + 1 \) and is induced \( P_m \)-saturated.

2.3 Other path results

For certain induced \( P_m \)-saturated graphs we can create induced \( P_{m+2} \)-saturated graphs by using the following constructions.

**Construction 2.12.** Generate the graph \( T_v(G) \) by identifying each vertex in \( G \) with one vertex of a distinct triangle. The new graph \( T_v(G) \) has order \( 3n(G) \) and size \( e(G) + 3n(G) \) (Figure 4).
Construction 2.13. The graph $T_v(G)$ is composed of the graph $G$ along with a new vertex for each edge of $G$, adjoined to both endpoints of that edge. The graph $T_v(G)$ has order $n(G) + e(G)$ and size $3e(G)$ (Figure 4).

Now we will show that both constructions yield the expected results. First, we restate and prove Theorem 2.14 in a different form than that given in Section 1.1.

Theorem 2.14. The graph $T_v(G_k)$ is induced $P_{11+6k}$-saturated.

Proof. First we establish that every vertex $v$ in the graph $G_k$ is the terminal vertex for two induced paths of order $8+6k$, each with a different terminal edge. Let $P$ be the path that begins $\{v_0v_5v_6v_7v_2v_3v_{n-2}v_{n-3}v_{n-4}v_{n-5}v_{n-6}v_{n-7}v_{n-12}\}$. If $i = (n-12)$ then proceed similar to $C$ in Fact 2.1 by adding $\{v_{i}v_{i-6}v_{i-5}v_{i-4}v_{i-3}v_{i-8}\}$ to $P$, repeating until the addition of edge $v_{10}v_{9}$. By applying the automorphism $\phi^8 \circ \psi$ we get another induced path of order $8 + 6k$ starting at $v_{0}$, $P'$. Notice that $P$ contains the edge $v_{0}v_{5}$ and $P'$ the edge $v_{0}v_{n-1}$. Again, since $G_k$ is vertex transitive, we see that each vertex in $G_k$ is the terminal vertex for two induced paths with distinct terminal edges.

Next, we consider a pair $x, y$ of distinct non-adjacent vertices in $G_k$. We need only consider the cases in which a pair of nonadjacent vertices are both in the original graph $G_k$, neither in the original graph $G_k$, or exactly one is in $G_k$.

Say that $x, y \in G_k$. Since the addition of edge $xy$ to $G_k$ creates an induced $P_{9+6k}$, one vertex from each added triangle to the endpoints of this path in $T_v(G_k)$ yields an induced $P_{11+6k}$. If neither $x$ nor $y$ are in $G_k$ and their neighbors in $G_k$, say $x'$ and $y'$ respectively, are not adjacent then, since a new edge between $x'$ and $y'$ in $G_k$ generates an induced $P_{9+6k}$, this path is extended similarly by two edges to create an induced $P_{11+6k}$ in $T_v(G_k)$. If instead $x'y'$ is an edge of $G_k$, consider the induced $P_{8+6k}$ in $G_k$ that begins at $x'$ and avoids the edge $x'y'$. This extends to an induced $P_{9+6k}$ in $T_v(G_k)$. Since $x'y'$ is not in this path, then the vertex $y'$ is also avoided entirely. The addition of edge $xy$ to $T_v(G_k)$ creates an induced $P_{11+6k}$ beginning with $y$.

Lastly, consider the case in which $x$ is a vertex of $G_k$ and $y$ is not. Let $y'$ be $y$’s neighbor in $G_k$ and $y''$ the vertex of degree 2 adjacent to $y$ in $T_v(G_k)$. Again, since there is an induced
$P_{9+6k}$ in $T_v(G_k)$ that begins at $x$ and avoids $y'$, the addition of the edge $xy$ creates an induced $P_{11+6k}$ beginning at $y''$.\[\square\]

**Theorem 2.15.** If $G$ is $K_3$-free, induced $P_m$-saturated, and every vertex is in a component of order at least three, then $T_e(G)$ is induced $P_{m+2}$-saturated.

**Proof.** Just as in the proof of Theorem 2.12 we need to consider the same three cases.

If $x, y$ are nonadjacent vertices, both in $G$, then the addition of edge $xy$ to $G$ generates an induced $P_m$. Since none of the neighbors of the end vertices are in the path, it can be extended on both ends to the added vertices, yielding an induced $P_{m+2}$. If both $x$ and $y$ are new vertices added in the construction of $T_e(G)$, then let the neighbors of $x$ be $x', x''$ and the neighbors of $y$ $y', y''$. If adding edge $x'y'$ or $x''y''$ creates an induced $P_m$ that avoids the edges $x'x''$ and $y'y''$ then the addition of edge $xy$ generates an induced $P_{m+2}$. If instead every induced $P_m$ created by adding either edge $x'y'$ or $x''y''$ includes at least one of the edges $\{x'x'', y'y''\}$ then there is an induced $P_m$ that includes the added edge $x'y''$ that does not since $G$ is $K_3$-free. Therefore the addition of edge $xy$ is equivalent to adding an edge between a neighbor of each in $G$, with the induced $P_m$ extended by one edge toward $x$ and one toward $y$. Lastly, if $x$ is in the original graph $G$ and $y$ is not, then we proceed as above and extend the induced $P_m$ that results from joining $x$ to a neighbor of $y$ not already adjacent to $x$ (which exists since $G$ is $K_3$-free) by one edge toward $y$ and by another edge at a terminal vertex of the path. Therefore, $T_e(G)$ is induced $P_{m+2}$-saturated. \[\square\]

We end this section by noting that a computer search using SAGE [11], in conjunction with Constructions 2.12 and 2.13, has found induced $P_m$-saturated graphs for all $7 \leq m \leq 30$. The results are listed in Table 5, most in LCF notation. In the interest of space we have omitted the proof that they are saturated, as each is simply a case analysis. Note that there are induced $P_m$-saturated graphs that cannot be written in the form $[x, -x]^n$, and further some are the result of the operations $T_e$ and $T_v$. Therefore, not all induced $P_m$-saturated graph are regular nor the result of a regular graph joined to a complete graph.

### 3 Cycles

The star $K_{1,(n-1)}$ is induced $C_3$-saturated, and is in fact the graph on $n$ vertices of smallest size for $n \geq 3$. This is a direct consequence of Theorem 1.1. The largest such graph is $K_{\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor}$ due to Mantel [9].

Note that $C_5$ is trivially both induced $C_3$-saturated and induced $C_4$-saturated.

We now show that for all integers $k \geq 3, n \geq 3(k - 2)$ there is an induced $C_k$-saturated graph on $n$ vertices that is non-complete. We begin with another construction.

**Construction 3.1.** Define the graph $G[k]$ on $3k$ vertices, $k \geq 3$, in the following way. Let $v_0 v_1 \ldots v_{k-1} v_0$ be a $k$-cycle, the internal cycle of $G[k]$. Add the matching $u_i v_i$ and the edges $u_i v_i, w_i v_i$, and $w_i u_{i+1}, 0 \leq i \leq (k - 1)$ with addition modulo $k$, the external cycle (Figure 6).
<table>
<thead>
<tr>
<th>$m$</th>
<th>induced $P_m$-saturated graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>$[4, -4, 5, -5]^3$</td>
</tr>
<tr>
<td>8, 9</td>
<td>$[5, -5]^7$</td>
</tr>
<tr>
<td>10</td>
<td>$[4, 6]^{10}$</td>
</tr>
<tr>
<td>11</td>
<td>$T_e([5, -5]^7])$</td>
</tr>
<tr>
<td>12, 13</td>
<td>$[5, -5, 9, -9]^5$</td>
</tr>
<tr>
<td>14, 15</td>
<td>$[5, 9]^{13}$</td>
</tr>
<tr>
<td>16</td>
<td>$[9, 15]^{12}$</td>
</tr>
<tr>
<td>17</td>
<td>$T_e([5, 9]^{13})$</td>
</tr>
<tr>
<td>18</td>
<td>$[5, 17]^{15}$</td>
</tr>
<tr>
<td>19, 20</td>
<td>$[7, 23]^{15}$</td>
</tr>
<tr>
<td>21, 22</td>
<td>$[5, -5, 13, -13]^8$</td>
</tr>
<tr>
<td>23</td>
<td>$[-17, 9]^{20}$</td>
</tr>
<tr>
<td>24</td>
<td>$T_e([5, -5, 13, -13]^8)$</td>
</tr>
<tr>
<td>25, 26</td>
<td>$[-15, 15]^{19}$</td>
</tr>
<tr>
<td>27</td>
<td>$[-13, 13]^{19}$</td>
</tr>
<tr>
<td>28</td>
<td>$[-15, 15]^{21}$</td>
</tr>
<tr>
<td>29, 30</td>
<td>$[-13, 13]^{22}$</td>
</tr>
</tbody>
</table>

Figure 5: Induced path-saturated graphs

Figure 6: The graph $G[5]$, which is induced $C_7$-saturated
Claim 3.2. The graph $G[k]$ is induced $C_{k+2}$-saturated.

Proof. First we show that $G[k]$ does not contain an induced cycle of length $k + 2$. Note that every copy of $C_{k+2}$ in $G[k]$ contains vertices from both the internal and external cycles. Any induced cycle $C$ in $G[k]$ contains paths on the internal cycle of the form $v_i v_{i+1} v_{i+2} \ldots v_j$ and/or paths on the outer cycle, and edges joining these paths into a cycle. The cycle $C$ therefore has length either $k$ (if $i = j$) or at least $(k+3)$.

Now we show that the addition of any edge $e$ to $G[k]$ results in an induced $C_{k+2}$. Note that there are three potential forms that $e$ can take: an edge among the vertices of the internal cycle, an external cycle edge, or an edge between these cycles. If $e = v_i v_j$ then we need only consider $k > 3$. There is a newly created induced cycle of length $l \geq \left(\left\lceil \frac{k}{2} \right\rceil + 1\right)$ along the internal cycle. This can be extended by considering an edge from $v_i$ to one of its neighbors on the external cycle, and traversing an appropriate number of edges before rejoining the internal cycle. In this way we create an induced cycle of every length between $(\left\lceil \frac{k}{2} \right\rceil + 4)$ and $(k+3)$, inclusive.

If instead $e$ joins vertices between the internal and external cycles, then we create an induced $C_{k+2}$ in the following way. Without loss of generality we assume that $e = v_i u_0$. We get an induced $C_{k-i+2}$ by proceeding around the internal cycle from $v_i$ to $v_{k-1}$ then to $u_{k-1}$. Other induced cycles result from returning to the outer cycle sooner, creating cycles of length $(k - i + 3)$ through $(2k - 2i + 1)$. We can also find induced cycles proceeding in the other direction along the internal cycle from $v_i$ down to $v_1$ (or $v_0$ if $i \neq (k - 1)$), then back to $u_1$, yielding induced cycles with lengths from $(i + 3)$ through $(2i + 2)$. Therefore, an induced cycle of length $(k+2)$ can be found in $G[k]$ with the added edge for $1 \leq i \leq \frac{k-1}{2}$ in the latter case and $\frac{k}{2} \leq i \leq (k - 1)$ in the former.

Finally, if the new edge joins vertices on the external cycle of $G[k]$ then an induced cycle of length $(k+2)$ can be formed by utilizing an edge from $v_i$ to the internal cycle, continuing along a sufficiently long path, then rejoining the described induced path along the external cycle.

Theorem 3.3. For any $k \geq 3$ and $n \geq 3(k-2)$ there is a non-complete induced $C_k$-saturated graph of order $n$.

Proof. By Claim 3.2 there is an $n$ and a graph $G[k-2]$ on $n$ vertices that is induced $C_k$-saturated. We can extend Construction 3.1 to a larger number of vertices by joining it to a clique, since any vertex in the joined clique is adjacent to all other vertices so cannot be in the induced cycle.

The graph $G[k]$ can also be extended to more vertices by replacing each vertex with a clique. If the vertices are distributed in a balanced way the resulting graph has size approximately $3k\left(\frac{n}{3k}\right)^2 + 5k\frac{n^2}{9k^2}$.
4 Claws

We now turn our attention to the claw graph $K_{1,3}$ (Figure 7). We build a graph that is induced $K_{1,3}$-saturated.

Construction 4.1. Let $G$ be a 6-cycle on the vertices $\{v_0, \ldots, v_5\}$. To this set, we add the vertices $\{u_0, \ldots, u_5\}$ and for each $i$ join $u_i$ to $v_i, v_{i+1},$ and $u_{i+3}$ with addition taken modulo 6 (Figure 7).

It is easy to see that the graph in Construction 4.1 is claw-free. We can now prove the following theorem.

Theorem 4.2. For all $n \geq 12$, there is a graph on $n$ vertices that is induced $K_{1,3}$-saturated and is non-complete graph.

Proof. First we demonstrate that the graph $G$ in Construction 4.1 is induced $K_{1,3}$-saturated. If we join $u_i$ to $u_j$ then $u_i$ is the center of a claw with neighbors $u_j, u_{i+3},$ and $v_i$. If we join $u_i$ to $v_j$ then $v_j$ has pairwise nonadjacent neighbors $u_i, v_{j-1}$ or $v_{j+1},$ and either $u_j$ or $u_{j-1}$. Finally, if edge $v_i v_j$ is added to $G$ then $v_i$ is the center of a claw along with $v_j, u_i,$ and $u_{i-1}$.

Since the disjoint union of induced $K_{1,3}$-saturated graphs is itself induced $K_{1,3}$-saturated we can generate a graph on $n$ vertices with disjoint copies of $G$ and possibly a complete connected component.

5 Future Work

It would be interesting to find a smallest construction $G(m)$ that is induced $P_m$-saturated for all $m > 1$, or determine that no such construction exists. It is suspected that $G(m)$ has size $\frac{3}{2}n(G(m))$, but the largest such graph, in the spirit of Turán’s Theorem [12], would also be worth investigating. Induced $P_m$-saturated graphs with pendant edges also remain to be studied, as these graphs may be smaller than those in Theorem 2.11. Indeed, it is not hard to construct such a graph by joining $K_1 \cup G_k$ to a single vertex, but this graph is quite large.
Further, as we have considered paths and claw graphs in this paper the study of induced saturation could be furthered by considering the family of trees.

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References


