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Tamara Pearson

Clayton State University, tamara.pearson@ceismc.gatech.edu

Kelli L. Nipper

Clayton State University, kellinipper@clayton.edu

Catherine Matos

Clayton State University, catherinematos@clayton.edu

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Do You See What I See? Deepening Teachers' Understanding of Linear Equations
Through Student Interviews

Tamara Pearson	Kelli Nipper	Catherine Matos
Clayton State University	Clayton State University	Clayton State University

Abstract

Many teachers have trouble transitioning their students between natural recursive thinking about the data and algebraic notation for representing linear functions (Zazkis & Liljedahl, 2002). In this study, we interviewed eighteen middle school students to see how they used prior instruction to think about a geometric pattern and construct its corresponding linear equation. All students were given the same task to complete and were questioned about their thinking during the interview.

We found that the recording of pattern recognition plays a substantial part in helping students recognize and write explicit patterns. By having students decompose the total perimeter into how they saw the pattern growing, students were more successful in making the connection to the numeric representation of growth. In addition, they were better able to explain how they set up the equation, and the connection of each part of the equation to the original pattern.

As teachers work with their students in developing a conceptual understanding of linear equations, it is critical that students are exposed to geometric patterns. The results of this study will help mathematics teacher educators better prepare teachers to develop their students' develop rich and connected mathematical understanding.

Keywords: Teacher Education; Conceptual Knowledge; Generalization; Linear Equations

Do You See What I See? Deepening Teachers' Understanding of Linear Equations
Through Student Interviews

Generalization of growth patterns, including visual patterns, is at the heart of algebraic thinking. (Blanton & Kaput, 2005; Friel & Markworth, 2009; Orton & Orton, 2004). The National Council of Teachers of Mathematics' (NCTM) Principles and Standards for School Mathematics (2000) states, "systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions" (p. 37). In addition, the newly adopted Common Core State Standards for Mathematics exposes students to growth patterns as early as kindergarten (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Looking at the hexagon series in Figure 1 below, and hypothesizing about how middle school students would approach finding a generalized equation for the perimeter's growth led us to wonder about the difficult process of helping students make the transition from recursive thinking to algebraic notation (Zazkis & Liljedahl, 2002a).

Figure 1. Hexagon series.

In the "Mathematics: Assessing Understanding" videos, Marilyn Burns (2010) conducted one-on-one interviews with elementary school students on number concepts they had previously been taught in class. She stated, "I've learned that one-on-one interviews reveal valuable information that is not available when I rely solely on students' written work" (p. 19). Many others

have explored similar questions about the creation of linear equations and have found the use of classroom settings and student interviews as a key component to the discovery of student conceptions. (Ma, 2007; Matsuura and Harless, 2012).

Based on these projects, we decided to interview students to understand their thought processes better as they attempted to generalize linear patterns. We felt that these interviews would not only allow us to explore the process of helping a student make the transition between the natural recursive thinking about the data and the more complex algebraic notation for representing the explicit function, but also to allow teachers the opportunity to identify critical aspects of instruction that play a important roles in students' recognition and writing of explicit patterns. Eighteen 6th through 8th grade students participated in 30-50 minute interviews. All students had recently received traditional instruction on the construction of linear equations and were all given the hexagon series (Figure 1) task to complete. Our goal for the interview was to see how students used that instruction to think about a geometric pattern and construct its corresponding linear equation.

In this article, we chose to focus on Janet. At the time she was interviewed, Janet was completing the 8th grade and had been identified by her teacher as above grade level in mathematics. Although other students may have struggled more with this task, Janet was selected because, in her interview, she clearly articulated most of the same student conceptions represented throughout all 18 interviews.

Janet's Method

When initially presented with the hexagon series task, Janet used an approach employed by most students. She counted the sides by hand for one and two tiles and then was able to make a prediction about the structure of the pattern for 3 tiles. She was then ready to attempt creating the corresponding linear equation that would represent the growth of this geometric pattern. In our work with teachers on tasks such as this, we have found that it is important that we emphasize

patience, and that they not rush their students to state a linear equation before students see the pattern for themselves. Janet was able to see it after two tiles, but other students took longer.

Janet's initial approach to determine the explicit linear equation was to focus on the components of slope and y-intercept in order to create a linear equation of the form $y=mx+b$, which is the traditional method taught in most classrooms. As she started this process, though, she was plagued by the fact that the pattern "added 4" each time.

Janet: I could say $4x$ because... Wait, no not $4x$, it would be plus 4. Something times x plus 4.

Interviewer: Tell me how you're getting that.

Janet: Cause I'm thinking about slope-intercept form. And it's $y=mx+b$.

Interviewer: And in that form what does the m represent?

Janet: Slope

Interviewer: And what does the b represent?

Janet: The y-intercept

Interviewer: So you said you think it's going to be y equals some number times x plus 4.

And tell me why you said plus 4.

Janet: Because in the pattern I saw that it's adding 4, so I thought about the plus part. So I said plus 4.

At this point Janet is struggling to make the link between her (correct) understanding of the pattern and her knowledge of how linear equations are structured. While Janet has no problem reciting the meaning of " m " and " b " in slope-intercept form, she demonstrates difficulty in trying to connect that knowledge to her intuition about the growth of the pattern. She knows this is the method she has been taught for constructing linear equations and wants to continue with it. After a brief discussion of slope and y-intercept and graphing the data she has gathered from the hexagon series, Janet's strong mathematical skills allow her to recognize her mistake and produce the correct

equation, $y=4x+2$. Follow-up questioning was used to determine her reasons for why the equation should include $4x$ and not plus 4 and highlighted her lack of conceptual understanding of the relationship between slope and rate of change.

Interviewer: Do you have any idea of why it is $4x$ instead of plus 4?

Janet: Because 4 is my slope.

Interviewer: So, why would we multiply by 4 instead of adding 4?

Janet: Because when I graphed it, the y-intercept isn't 4.

Janet's struggle is a typical example of the purely procedural level of understanding of the construction of linear equations of many of the students we interviewed. The students demonstrated their inability to interpret the repeated addition of four as multiplication by four in the final equation (Ma, 2007; Zazkis & Liljedahl, 2002a, 2002b). Presented with patterning tasks, such as arithmetic sequences, with a set of data, the students were typically capable of determining the correct linear equation to model the data. When asked about the meaning of slope and y-intercept, middle school students had little problem reciting memorized definitions from their textbooks. However, when they were asked to connect those definitions to a more conceptual understanding of growing patterns, they floundered (Markworth, 2012; Rivera & Rossi Becker, 2005).

Moving Towards Conceptual Understanding

Building on the outcomes of this interview, our goal for Janet became to help her to progress from a strictly procedural understanding of linear equations to a conceptual one. Many who have written previously about geometric patterns have outlined strategies for facilitating this progression in students (Friel & Markworth, 2009; Lee & Freiman, 2006; Markworth, 2012; Rivera & Rossi Becker, 2005). These strategies all contain the same key components, the idea that in order for students to construct explicit equations based on visualization of patterns they must (1) see the pattern, (2) appropriately document the pattern, and (3) make the connection between what they see and document to an algebraic representation. Our interviews were guided by the desire to help

students to learn how to employ these strategies.

Seeing the Pattern. We began the process by refocusing Janet on the original geometric pattern. This time, however, we emphasized having her explain how she sees the perimeter growing, rather than just the measure of the perimeter at each step.

Interviewer: Is there any other way that you see 10 in this group of 2 tiles?

Janet: (thinking) Since it's a hexagon, you know the whole thing around is gonna be 6.

When you put them together it gives you 12, but then you have to take off two sides cause you connected it.

Interviewer: So let's write that, what you see for two tiles is 6 plus 6, and then what happens after that?

Janet: Then you have to take away 2 sides.

Interviewer: So that would be minus 2. Excellent. So for three tiles what do you see happening?

Janet: You do the same thing. Six plus 6 plus 6, but this time you subtract 3.

Interviewer: Can you show me those three?

Janet: (counts the sides with her hands) Actually it's 4.

Interviewer: So let's go ahead and write that. What about for 4 tiles?

Janet: (checks her answer by counting the subtracted sides on the tiles). Six plus 6 plus 6 plus 6 minus 6.

Based on Janet's verbal explanation of her thinking, Figure 2 shows a visual representation of the manner in which Janet described the pattern growing.

Figure 2. A visual representation of Janet's thinking.

Janet was able to use her improved understanding of the visual nature of the pattern growth to connect to her understanding of the numeric representation of growth. The interviewer was then able to help her understand how to represent that thinking mathematically (Friel & Markworth, 2009; Ma, 2007). Guiding their students through this step is often difficult for teachers who have never explored patterns using a visual approach. Therefore, as teacher educators, we must expose pre-and in-service teachers to pattern exploration activities and challenge them to see and represent patterns in as many different ways as possible, not just using the traditional slope-intercept method (Zazkis & Liljedahl, 2002b).

Recording the Pattern. The importance of teachers guiding students through the mathematical recording of their findings, with a particular focus on connecting the visual understanding of the pattern and a corresponding algebraic representation, has been illustrated by research (National Council of Teachers of Mathematics, 2000; Zazkis & Liljedahl, 2002b). Without this connection, it can be difficult for students to find patterns within their own work. Figure 3 shows Janet's notes on the construction of the pattern. Due to the method in which she documented the pattern, she was able to quickly discern the pattern's structure.

$$\begin{array}{l} 2 \text{ tiles} = 6 + 6 - 2 \\ 3 \text{ tiles} = 6 + 6 + 6 - 4 \\ 4 \text{ tiles} = 6 + 6 + 6 + 6 - 6 \end{array}$$

Figure 3. Janet's initial notes connecting her visual representation to the perimeter.

Determining the amount to subtract proved to be the most difficult task for Janet. Since she had documented the pattern, she was able to connect that the number of groups of 6 was the same as the number of hexagon tiles. Seeing the pattern in the subtraction part was a struggle for Janet, though, since the amount subtracted was different for each set of tiles. When the interviewer questioned her about how much she would need to subtract for nine tiles, she initially used a recursive strategy, counting up, by 2, from 4 tiles. With this approach, she was able to determine that for nine tiles you would subtract 16. This method is employed by many students and is referred to by Kaye as “near and far generalizations” (1989, p. 155). Through her research, Kaye found that when students were presented with a growth pattern, they typically used the same strategies for extending the pattern to the 20th or 100th value as they had from the 2nd to 3rd value. The interviewer was then able to use Janet's recursive thinking to help her to connect it to an explicit linear equation.

Even though Janet knew that the amount subtracted from the total perimeter increased by two with each additional tile, and she used that understanding to “count up” using a recursive model, she was not documenting her work to reflect how the amount changes each time. Figure 3 shows how Janet documents only the total amount subtracted for each pattern, not how that total was constructed. To help her uncover patterns, the interviewer encouraged Janet to be more specific in recording of her thinking. Her revised documentation method in Figure 4 shows that Janet was not originally noting the groups of two. Through conversation with the interviewer, she added a superscript notation to her work to keep track of how many groups of two comprised the

total amount subtracted each time. From this notation, Janet was able to determine the pattern in the groups of two and extend this discovered pattern to document the perimeter of 100 tiles.

$$\begin{aligned}
 2 \text{ tiles} &= 6 + 6 - 2^1g \\
 3 \text{ tiles} &= 6 + 6 + 6 - 4^2g \\
 4 \text{ tiles} &= 6 + 6 + 6 + 6 - 6^3g \\
 9 \text{ tiles} &= 6 \times 9 - 16^8g \\
 20 \text{ tiles} &= 6 \times 20 - 38 = 82^19g \\
 100 \text{ tiles} &= 6 \times 100 - 99 \times 2 =
 \end{aligned}$$

Figure 4. Janet's revised notes connecting her visual representation to the perimeter.

Connecting the Pattern to an Algebraic Representation. The final step in the process was for Janet to extend this specific 100-tile example to a general linear equation to determine the perimeter for any number of tiles. This extension is one that many students that we interviewed struggled with, though it may seem to be a trivial step. Janet's final equation, shown in Figure 5, does not look anything like the one she generated at the start of the process using slope-intercept form ($y=4x+2$). This difference often causes students to struggle with developing linear equations. What they are seeing and understanding in their minds about the pattern does not match their limited understanding of linear equations, so they don't understand how the two representations are connected (Kaye, 1989; Orton & Orton, 1999). If Janet simplifies this equation, she will see that it is exactly the same as the one she created in the beginning of the interview.

$$X = 6x - (x - 1) \times 2$$

Figure 5. Janet's final equation.

Conclusions and Implications for Teacher Education

Much of the research on the use of student interviews focuses on student learning. In this investigation, we were able to look carefully at student learning but with an eye on how that learning impacts teacher education. For many of the students interviewed, including Janet, the

process wasn't only about developing a linear equation for the hexagon pattern. Janet was able to do that with minimal assistance early in the interview. The larger purpose was for the interviewer/teacher to better understand how Janet and the other students think about patterns, how they make connections across representations, and for students to experience a different perspective on linear equations.

From this case study, we have found that the recording of the pattern plays a substantial role in helping students recognize and write explicit patterns. By decomposing the perimeter, students were more successful in abstracting the patterns. In addition, they were better able to explain how they developed the equation, and how they connected each part of the equation to the original pattern.

To prepare our future teacher candidates to utilize this approach in their classes, we include a “rule” column in the data table between the input and output columns for geometric pattern tasks, including linear, quadratic and exponential functions. The “rule” column is meant to provide a workspace for students to record their pattern in the way they have visualized the pattern growth. For example, for Janet, the completed table would look like the one shown in Table 1.

Number of Hexagons	“RULE”	Perimeter
1	6	6
2	$6+6-2$	10
3	$6+6+6-2-2$	14

Table 1. Janet's pattern using the “rule” column.

One advantage that we have recognized from including the workspace in the table is that the pictorial, verbal and table representations are better connected. In the pictorial representation, students build and draw the next two figures. In the verbal representation, students describe how they see the perimeter growing. In the table, they record the pattern in the way they recognized the

pattern growing before finding the total perimeter. Finally, they create the equation based on the decomposed rule in the table connecting each component of the equation to the original pattern. (See Appendix)

An additional consequence of including the “rule” column is that teachers become more aware that the way they personally see the pattern grow is not the way that everyone sees it. Therefore, they are more likely to recognize multiple growth patterns for a given task. In Janet’s case, she recognized the growth in terms of each hexagon having six sides with some sides needing to be subtracted due to the matching up of sides. Another standard approach is for students to visualize the perimeter growing by focusing on the tops and the bottoms and then adding the two ends. Recognizing that there are multiple ways to see pattern growth is an important step in being able to help students transition between the recursive, additive nature of linear growth patterns and the traditional algebraic notation for representing linear equations (Zaskis & Liljedahl 2002a).

As teachers work with students in developing conceptual understanding of linear equations it is critical that they expose students to geometric patterns (Friel & Markworth, 2009). Linear equations are the first growing patterns that many students experience, but not the last. Using the methods discussed in this article will better prepare teachers to understand the complexity of helping their students develop rich and connected mathematical understanding. Specifically, we recommend that mathematics teacher educators provide many opportunities for teachers to experience the connections between the pictorial, verbal, table, and equation representations of geometric patterning tasks. As teachers deepen their understanding of the connection of each representation to the original pattern, they will be better prepared to help students develop a conceptual understanding of linear equations.

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Appendix

Hexagon Perimeter Task

What “rule” describes the perimeter of the hexagon trains given any number of hexagons?

(1)

(2)

(3)

PICTORIAL: Build and then draw the next two figures in the train.

VERBAL: Describe how you see the perimeter growing as the number of hexagon tiles increase?

TABLE: Complete the table below.			EQUATION: Write an equation using SYMBOLS for finding the perimeter of a row of hexagonal tiles when you know the number of tiles in the row.
Number of Hexagons	“RULE”	Perimeter	
1			
2			
3			
4			