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Eigenvalue Estimates of Laplacians Defined by Fractal Measures

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Definition of Dirichlet Laplacian: weak formulation

Assumptions throughout this talk:

(1) $\Omega \subseteq \mathbb{R}^n$ bounded, open, connected.

(2) $\mu =$ regular Borel probability measure on $\mathbb{R}^n$, $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$, $\mu(\Omega) > 0$.

Poincaré type inequality:

(PI) $\exists$ constant $C > 0$ s.t.

$$\int_\Omega |u|^2 \, d\mu \leq C \int_\Omega |\nabla u|^2 \, dx \quad \forall u \in C_c^\infty(\Omega). \tag{1.1}$$

Elements in the same equivalence class of $H_0^1(\Omega)$ can belong to different equivalence classes of $L^2(\Omega, \mu)$. (PI) $\Rightarrow$ each equivalence class of $u \in H_0^1(\Omega)$ contains a unique $\bar{u} \in L^2(\Omega, \mu)$ satisfying (1.1).

Define $\iota : H_0^1(\Omega) \to L^2(\Omega, \mu)$, $\iota(u) = \bar{u}$. Let $\mathcal{N} := \ker \iota$. Then $\iota : \mathcal{N}^\perp \hookrightarrow L^2(\Omega, \mu)$.

Define a nonnegative bilinear form $\mathcal{E}(\cdot, \cdot)$ on $L^2(\Omega, \mu)$ by

$$\mathcal{E}(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx,$$

with $\mathrm{Dom}(\mathcal{E}) = \mathcal{N}^\perp$ (more precisely $\iota(\mathcal{N}^\perp)$).

Facts: Assume (PI).

(a) Then $\mathcal{E}$ is closed. Denote the corresponding self-adjoint operator by: Δ_μ, *Dirichlet Laplacian* with respect to μ.

(b) Let $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega, \mu)$. TFAE:

(i) $u \in \mathrm{Dom}(\Delta_\mu)$ and $\Delta_\mu u = f$;

(ii) $\Delta u = f\, d\mu$ as distributions.

$\Delta_\mu(u) = u_{tt}$ models a (nonhomogeneous) vibrating string or membrane with mass distribution μ.

A sufficient condition for (PI)

Definition: The *lower L^∞-dimension* of μ:

$$\underline{\dim}_\infty(\mu) = \varliminf_{\delta\to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta}.$$

where the supremum is taken over all $x \in \operatorname{supp}(\mu)$.

Based mainly on a result of Maz'ja, we obtained:

Theorem

(Hu-Lau-N., 06) Assume that $\underline{\dim}_{\infty}(\mu) > d - 2$.

(a) *(PI) holds.*

(b) $\exists$ *an orthonormal basis* $\{u_n\}_{n=1}^{\infty}$ *of* $L^2(\Omega, \mu)$ *consisting of eigenfunctions of* $-\Delta_{\mu}$.

(c) *The eigenvalues of* $\{\lambda_n\}_{n=1}^{\infty}$ *satisfy*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \qquad \textit{with} \qquad \lim_{n\to\infty} \lambda_n = \infty.$$

Motivations:

1) Estimate the Poincaré constant $= \frac{1}{\lambda_1^\mu}$.
2) Study existence of spectral gaps.
3) Study of bounds for eigenfunctions.

Classical results:

- Faber-Krahn, 23, 25: Let $\Omega \subset \mathbb{R}^n$ and $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B_r(0))$.

$$\lambda_1(\Omega) \geq \lambda_1(B_r(0)).$$

Not true for manifolds!

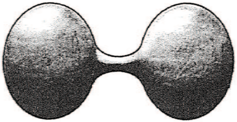


Figure: E. Calabi: dumbbell M homeomorphic to S^2, $\lambda_1 \to 0$ as radius of connecting pipe (of fixed length) $\to 0$.

- Cheeger, 70: M be a compact Riemannian manifold.

$h_D(M) := \inf \left\{ \dfrac{\mathrm{Vol}(\partial U)}{\mathrm{Vol}(U)} : U \subset\subset M \right\}$. Then

$$\lambda_1 \geq \frac{1}{4} h_D^2(M).$$

Theorem 1. *Let* $\Omega = (0,1)$. *Then* $\lambda_1^\mu \geq \pi$.

We improve this theorem for the infinite Bernoulli convolutions:

$$S_1(x) = rx, \quad S_2(x) = rx + 1 - r, \quad 0 < r < 1,$$

Let $\mu = \mu_{r,p}$ be the associated self-similar measure:

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}, \quad 0 < p < 1.$$

$\mathrm{supp}\mu \subseteq [0,1]$.

Proposition 2. *Let μ be an infinite Bernoulli convolution on $[0,1]$ with $p = 1/2$. Then $\lambda_1^\mu > \pi$. In fact,*

(a) *for $r \in (0, 1/2]$ (μ is a Cantor-type or Lebesgue), $\lambda_1^\mu \geq 4/r$;*

(b) *for $r \in (1/2, 2/3]$, $\lambda_1^\mu \geq 24/7 \approx 3.428$;*

(c) *for $r \in (2/3, 1)$, $\lambda_1^\mu \geq 3.2$.*

Definition

μ on $\Omega \subseteq \mathbb{R}^n$ is upper s-regular *if $\exists c_1 > 0$ s.t.*

$$\mu(F) \leq c_1 |F|^s \quad \text{for all } \mu \text{ measurable subsets } F \subseteq \Omega.$$

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$. Suppose that μ is upper s-regular for some $s \geq 1$. Let $\beta = c_1 |\Omega|^{s-1}/2$ with c_1 being the constant above. Then*

$$\lambda_1^\mu \geq \frac{\sqrt{\lambda_1}}{2\beta}.$$

where λ_1 is the first eigenvalue of $-\Delta$.

$\Omega \subset \mathbb{R}^n$ and μ absolutely continuous. By using Cheeger's argument:

Theorem 4. *Let $\Omega \subseteq \mathbb{R}^n$, μ be absolutely continuous w.r.t. Lebesgue measure with a bounded density ρ, and*

$$\beta := \inf_{U \subseteq \Omega} \frac{1}{\mu(U)} \int_{\partial U} \rho \, d\mathcal{H}^{n-1}.$$

Then $\lambda_1^\mu \geq \beta^2 / (4\|\rho\|_{L^\infty(dx)})$.

Upper estimates

It is not clear how to obtain upper estimates in general.
We can obtain upper estimates for infinite Bernoulli convolution

1) Cantor type $r < 1/2$, and
2) golden ratio $(\sqrt{5} - 1)/2$.

- Erdős, 39: μ is singular (also true for other Pisot numbers).
- L^q-spectrum, multifractal formalism, and dimension (Lau-N., 98, 99, Feng 05).
- Spectral dimension (N., 11).
- Wave equation (Chan-Teplyaev-N., to appear).

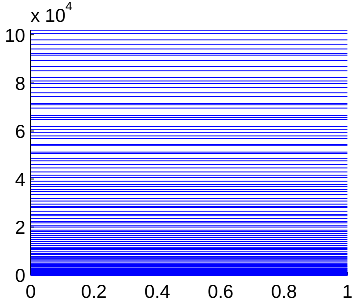
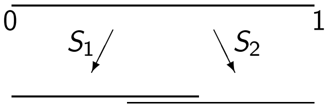
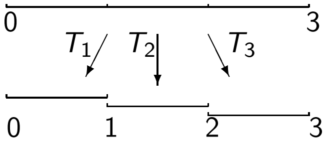


Figure: First 100 Dirichlet eigenvalues for the Bernoulli convolution associated with the golden ratio. (Chen-N., 10)



Second-order identities by Strichartz-Taylor-Zhang, 95:

$$\begin{aligned} T_1(x) &= S_1S_1(x) = r^2x, \\ T_2(x) &= S_1S_2S_2(x) = S_2S_1S_1(x) = r^3x + r^2, \\ T_3(x) &= S_2S_2(x) = r^2x + r. \end{aligned}$$



For any Borel subset $A \subseteq [0,1]$,

$$\begin{bmatrix} \mu(T_1T_jA) \\ \mu(T_2T_jA) \\ \mu(T_3T_jA) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad j = 1,2,3,$$

where M_1, M_2, M_3 are, respectively,

$$\begin{bmatrix} p^2 & 0 & 0 \\ (1-p)p^2 & (1-p)p & 0 \\ 0 & 1-p & 0 \end{bmatrix}, \begin{bmatrix} 0 & p^2 & 0 \\ 0 & (1-p)p & 0 \\ 0 & (1-p)^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & p & 0 \\ 0 & (1-p)p & (1-p)^2p \\ 0 & 0 & (1-p)^2 \end{bmatrix}.$$

Let $J = j_1 \cdots j_m$, $j_i \in \{1,2,3\}$. Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad \text{where} \quad c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

Define

$$w_J^* := \mu(T_J[0,1]) = \frac{1}{1-p+p^2} c_J \begin{bmatrix} p^2 \\ p(1-p) \\ (1-p)^2 \end{bmatrix}.$$

Theorem 5. *Let $\mu = \mu_{r,p}$ be an infinite Bernoulli convolution and f_m be a piecewise linear approximate first eigenfunction obtained by the finite element method.*

(a) *If $0 < r \leq 1/2$, then*

$$\lambda_1^{\mu_{r,p}} \leq \frac{\sum_{J\in\mathcal{J}_m}(f_m(T_J(1)) - f_m(T_J(0)))^2/(T_J(1) - T_J(0))}{\sum_{J\in\mathcal{J}_m} w_J \min\{f_m(T_J(1))^2, f_m(T_J(0))^2\}},$$

for all $m \geq 1$. For the standard Cantor measure, we have

$$12 \leq \lambda_1^{\mu_{1/3,1/2}} \leq 14.3865.$$

(c.f.: numerical approximation by FEM with $m = 7$ is $14.4353\ldots$)

(b) If $r = (\sqrt{5}-1)/2$, the reciprocal of the golden ratio, then

$$\lambda_1^{\mu_{r,p}} \leq \frac{\sum_{J\in\mathcal{J}_m}(f_m(T_J(1)) - f_m(T_J(0)))^2/(T_J(1) - T_J(0))}{\sum_{J\in\mathcal{J}_m} w_J^* \min\{f_m(T_J(1))^2, f_m(T_J(0))^2\}}$$

for all $m \geq 1$. In particular, if $p = 1/2$, we have

$$6.33437 \leq \lambda_1^{\mu_{r,1/2}} \leq 8.05171.$$

(c.f.: numerical approximation by FEM with $m = 7$ is $8.03475\ldots$)

Preparations for proofs.

Definition

Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded), $\emptyset \neq K \subset \Omega$ compact. Define the 1-capacity *of K relative to Ω:*

$$\mathrm{Cap}_1(K,\Omega) := \inf\left\{ \int_\Omega |\nabla\psi|\, dx : \psi \in C_c^\infty(\Omega), \psi \geq 1 \text{ on } K \right\}.$$

Examples:

1) $K \subset \mathbb{R}$ compact $\quad \Leftrightarrow \quad \mathrm{Cap}_1(K,\mathbb{R}) = 2.$

2) $\mathrm{Cap}_1(K,\mathbb{R}^n) = 0 \quad \Leftrightarrow \quad \mathcal{H}^{n-1}(K) = 0.$

Proposition

(Maz'ja, Sobolev spaces) Let $\Omega \subseteq \mathbb{R}^n$. If $\exists$ a constant $\beta > 0$ s.t. $\forall$ compact $F \subset \Omega$,

$$\mu(F) \leq \beta \, \mathrm{Cap}_1(F, \Omega),$$

then

$$\int_\Omega |u| \, d\mu \leq \beta \int_\Omega |\nabla u| \, dx, \forall u \in C_c^\infty(\Omega),$$

Bound of μ-measure by 1-capacity $\longrightarrow$ 1-Poincaré inequality.

We also need Rayleigh's formula:

$$\lambda_1^\mu = \inf_{f \in \mathrm{Dom}\mathcal{E}} \frac{\int_\Omega (f')^2 \, dx}{\int_\Omega f^2 \, d\mu}.$$

Proof of Theorem 1:

Step 1. By Rayleigh's formula for λ_1,

$$\lambda_1 \le \frac{\int_0^1 (f')^2\,dx}{\int_0^1 f^2\,dx}, \qquad \lambda_1^\mu = \frac{\int_0^1 (f')^2\,dx}{\int_0^1 f^2\,d\mu}.$$

Hence

$$\frac{\lambda_1^\mu}{\lambda_1} \ge \frac{\int_0^1 f^2\,dx}{\int_0^1 f^2\,d\mu}. \tag{3.2}$$

Step 2. For $F \subset \Omega$, $\mu(F) \leq 1$ and $\mathrm{Cap}_1(F, \mathbb{R}) = 2$ implies

$$\mu(F) \leq \mathrm{Cap}_1(F, \mathbb{R})/2.$$

Thus by the Proposition, For all $u \in H_0^1(\Omega)$,

$$\int_0^1 |u| \, d\mu \leq \frac{1}{2} \int_0^1 |u'| \, dx \quad \forall u \in C_c^\infty(\Omega). \tag{3.3}$$

For $u = [\bar{u}] \in H_0^1(\Omega)$, take $\{u_n\} \in C_c^\infty(\Omega)$ that converges to $\bar{u}$ simultaneously in $H_0^1(\Omega)$ and $L^2(\Omega, \mu)$. Then $\mu(\Omega) < \infty$ implies that $u_n \to \bar{u}$ in $L^1(\Omega, \mu)$ and $\nabla u_n \to \nabla \bar{u}$ in $L^1(\Omega, dx)$. Taking limit.

Step 3. (Cheeger's argument: Convert L^1-estimate to L^2-estimate.) Taking $u = f^2$, we get

$$2\int_0^1 f^2\,d\mu \le \int_0^1 2|f||f'|\,dx \le 2\Big(\int_0^1 f^2\,dx\Big)^{1/2}\Big(\int_0^1 (f')^2\,dx\Big)^{1/2}.$$

Hence

$$\lambda_1^\mu \int_0^1 f^2\,d\mu = \int_0^1 (f')^2\,dx \ge \frac{(\int_0^1 f^2\,d\mu)^2}{\int_0^1 f^2\,dx},$$

implying

$$\lambda_1^\mu \ge \frac{\int_0^1 f^2\,d\mu}{\int_0^1 f^2\,dx}. \tag{3.4}$$

Combining (3.2) and (3.4) gives $(\lambda_1^\mu)^2 \ge \lambda_1 = \pi^2$.

Prove of Theorem 3 is similar.

Step 1: Use Rayleigh formula for λ_1.

Step 2: Prove that $\mu(F) \leq \beta \mathrm{Cap}_1(F;\Omega)$ (needs upper s-regularity) and thus by Proposition

$$\int_\Omega |u| \, d\mu \leq \beta \int_\Omega |\nabla u| \, dx.$$

Step 3: Cheeger's argument.

Step 2: Claim: μ upper s-regular $s \geq 1$ (i.e., $\mu(F) \leq c_1|F|^s$). Then for all compact $F \subset \Omega$,

$$\mu(F) \leq \beta \mathrm{Cap}_1(F, \Omega) \quad \forall \text{ compact } F \subset \Omega,$$

where $\beta = c_1|\Omega|^{s-1}/2$.

Reason: Suppose $F \subset \Omega$ compact and connected. F a singleton: then $\mu(F) = 0$ and conclusion holds trivially for any β. F not a singleton: its connectedness implies that $\mathrm{Cap}_1(F; \Omega) > 0$. Let ball $B_{|F|}$ contain F. Then $|F|^s = \pi^{-s/2}\mathcal{L}^2(B_{|F|})^s/2$. Notice:

$$\mathrm{Cap}_1(F; \Omega) \geq \mathrm{Cap}_1(F; \mathbb{R}^2) = \mathrm{Cap}_1(\mathrm{co}(F); \mathbb{R}^2) \geq 2|F|.$$

Hence

$$\frac{|F|^s}{\mathrm{Cap}_1(F;\Omega)} \leq \frac{\pi^{-s/2}(\mathcal{L}^2(B_{|F|}))^{s/2}}{2|F|} = \frac{|F|^{s-1}}{2} \leq \frac{|\Omega|^{s-1}}{2}.$$

Together with upper s-regularity, we get

$$\frac{\mu(F)}{\mathrm{Cap}_1(F;\Omega)} \leq \frac{C|\Omega|^{s-1}}{2}.$$

For general compact sets, approximate from outside by the union of countably many compact connected sets.

Main ideas in the proof of Proposition 2: (Use some results in Bird-Teplyaev-N. 2003)

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}.$$

Let u be an λ_1^{μ}-eigenfunction corresponding to with $u'(0) = 1$.
Known: u and u' are continuous and

$$u'(x) = u'(0) - \lambda_1^{\mu} \int_0^x u(y)\, d\mu(y).$$

Also, $u'(1/2) = 0$ by symmetry.

Hence

$$\lambda_1^\mu = \Big(\int_0^{1/2} u(y)\, d\mu(y) \Big)^{-1} = \Big(\int_0^{1-r} u(y)\, d\mu(y) + \int_{1-r}^{1/2} u(y)\, d\mu(y) \Big)^{-1}$$

1) Use concavity of u and $u'(0) = 1$ to get $u(y) \leq y \ \forall y \in [0,1]$.
2) Fact: $\int_0^1 y\, d\mu(y) = 1/2$.
3) Self-similar identity.

Proof of Theorem 4. Step 1. Suppose $\exists \beta > 0$ s.t.

$$\beta \int_\Omega |\phi| \, d\mu \le \int_\Omega |\nabla \phi| \, d\mu \quad \forall \ \phi \in C_c^\infty(\Omega) \text{ (hence } H_0^1). \qquad (3.5)$$

Then $\lambda_1^\mu \ge \dfrac{\beta^2}{4\|\rho\|_\infty}$.
Take $\phi = f^2$, f a λ_1^μ-eigenfunction. Assume $\phi \in H_0 1^(\Omega)$; otherwise, take truncations of f and then take limit.

$$\beta \int_\Omega f^2 \, d\mu \le 2 \int_\Omega |f||\nabla f| \, d\mu \le 2\|f\|_{L^2(\mu)} \|\nabla f\|_{L^2(\mu)},$$

which implies that

$$\frac{\beta^2}{4}\int_\Omega f^2\,d\mu \le \int_\Omega |\nabla f|^2\rho\,dx \le \|\rho\|_\infty \int_\Omega |\nabla f|^2\,dx.$$

Thus,

$$\frac{\beta^2}{4\|\rho\|_\infty}\int_\Omega f^2\,d\mu \le \int_\Omega |\nabla f|^2\,dx.$$

Notice that $\lambda_1^\mu \int_\Omega f^2\,d\mu = \int_\Omega |\nabla f|^2\,dx$.

Step 2. Proof of (3.5). Using the coarea formula,
Let ψ be a nonnegative Borel measurable function on Ω and let $u \in C^{0,1}(\Omega)$. Then

$$\int_{\Omega} \psi(x) |\nabla u(x)| \, dx = \int_{-\infty}^{\infty} \Big(\int_{\{|\psi|=t\}} \psi(x) \, d\mathcal{H}^{n-1}(x) \Big) \, dt.$$

In its simplest form $\psi \equiv 1$, $n = 2$, $u =$ "2-dim tent function", the formula says the area of a disk can be evaluated by integrating the circumferences of circles making up the disk.

$$\begin{aligned}
\int_\Omega |\nabla\phi|\,d\mu &= \int_\Omega |\nabla\phi|\rho\,dx = \int_0^\infty \Big(\int_{|\phi|=t} \rho\,d\mathcal{H}^{n-1}\Big)\,dt \\
&= \int_0^\infty \Big(\frac{1}{\mu\{|\phi|\geq t\}}\int_{\{|\phi|=t\}} \rho\,d\mathcal{H}^{n-1}\Big)\mu\{|\phi|\geq t\}\,dt \\
&\geq \int_0^\infty \Big(\frac{1}{\mu\{|\phi|\geq t\}}\int_{\partial\{|\phi|\geq t\}} \rho\,d\mathcal{H}^{n-1}\Big)\mu\{|\phi|\geq t\}\,dt \\
&\geq \left(\inf_{U\subseteq\Omega}\frac{1}{\mu(U)}\int_{\partial U}\rho\,d\mathcal{H}^{n-1}\right)\int_0^\infty \mu\{|\phi|\geq t\}\,dt \\
&= \beta\int_\Omega |\phi|\,d\mu
\end{aligned}$$

Theorem 4 proved.

Proof of Theorem 5.

- We choose any positive numerical first eigenfunction f_m.
- f_m is piecewise linear and $\int_0^1 (f_m')^2\,dx$ can be evaluated exactly.
- For $\int_0^1 f_m^2\,d\mu$, multiply the measure of an interval between two consecutive nodes by the minimum value of f_m on that interval to get a lower bound $s(f_m)$.
- Use Rayleigh's formula

$$\lambda_1^\mu = \min_{u\in\mathrm{Dom}(\mathcal{E})} \frac{\int_0^1 (u')^2\,dx}{\int_0^1 u^2\,d\mu} \le \frac{\int_0^1 (f_m')^2\,dx}{\int_0^1 f_m^2\,d\mu} \le \frac{\int_0^1 (f_m')^2\,dx}{s(f_m)}.$$

Problems for further study:

1) Other eigenvalues $\lambda_n^\mu, n \geq 2$.
2) Spectral gap conjecture.
3) Relationship between eigenvalues and geometry of the set and measure-theoretic properties of μ.

Thank you!