Theory and Applications of Graphs

Volume 2 | Issue 1 Article 2

2015

Second Hamiltonian Cycles in Claw-Free Graphs

Hossein Esfandiari
hossein@cs.umd.edu

Colton Magnant
Georgia Southern University, cmagnant@georgiasouthern.edu

Pouria Salehi Nowbandegani
pouria.salehi@gmail.com

Shirdareh Haghighi
shirdareh@susc.ac.ir

Recommended Citation

Esfandiari, Hossein; Magnant, Colton; Salehi Nowbandegani, Pouria; and Haghighi, Shirdareh (2015) "Second Hamiltonian Cycles in Claw-Free Graphs," *Theory and Applications of Graphs*: Vol. 2 : Iss. 1 , Article 2.
DOI: 10.20429/tag.2015.020102
Available at: https://digitalcommons.georgiasouthern.edu/tag/vol2/iss1/2

Abstract

Sheehan conjectured in 1975 that every Hamiltonian regular simple graph of even degree at least four contains a second Hamiltonian cycle. We prove that most claw-free Hamiltonian graphs with minimum degree at least 3 have a second Hamiltonian cycle and describe the structure of those graphs not covered by our result. By this result, we show that Sheehan's conjecture holds for claw-free graphs whose order is not divisible by 6. In addition, we believe that the structure that we introduce can be useful for further studies on claw-free graphs.

Keywords: claw-free graph; Sheehan's conjecture; second Hamiltonian cycle

1 Introduction

All graphs in this paper are assumed to be simple without loops or multi-edges. We use the notation of [2].

In [11], Cedric Smith proved that every edge of a 3-regular graph appears in an even number of Hamiltonian cycles, thereby implying that a Hamiltonian 3-regular graph contains a second Hamiltonian cycle. This result was later extended by Thomason in [9] to Hamiltonian d-regular graphs for all odd d. More generally, Thomason proved the following.

Theorem 1 ([9])**.** *If every vertex has odd degree, then there are an even number of Hamiltonian cycles through any edge.*

Sheehan (see [8] and [2, page 590, problem 83]) conjectured that regular Hamiltonian graphs of even degree at least four are also not uniquely Hamiltonian.

Conjecture 1 (Sheehan's Conjecture [8])**.** *If G is a Hamiltonian regular graph with even degree $d \geq 4$, then G contains a second Hamiltonian cycle.*

In fact, Sheehan [8] only conjectured this result in the case when $d = 4$. If one can prove the conjecture in 4-regular graphs, then it can be easily deduced for all larger even degrees using Petersen's Theorem [7]. Thus, Conjecture 1 can be restated as follows.

Conjecture 2. *If G is a Hamiltonian 4-regular graph, then G contains a second Hamiltonian cycle.*

Thomassen [10] showed Conjecture 1 holds for $d > 72$. This was later improved by Ghandehari and Hatami [5] who proved the conjecture for $d > 48$. The best known progress is due to Haxell, Seamone, and Verstraete [6], who settled the conjecture for degree $d > 22$.

Theorem 2 ([6])**.** *If G is a Hamiltonian regular graph with degree $d > 22$, then G contains a second Hamiltonian cycle.*

It should also be noted that related results were proven in [3] and [4] where constructions were provided of graphs with minimum degree 3 having no third Hamiltonian cycle and graphs with minimum degree 4 having no second Hamiltonian cycle respectively.

Recall that a graph containing no copy of a particular graph H as an induced subgraph is called *H-free* and the complete bipartite graph $K_{1,3}$ is referred to as a *claw*. Thus, a graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. In this work we study Sheehan's Conjecture 2 in claw-free graphs.

The Second Hamiltonian Cycle problem in claw-free graphs appears to be non-trivial and has been considered in previous literature. Bielak [1] showed that every Hamiltonian claw-free graph with minimum degree at least 3 contains an edge whose deletion and contraction each yield a Hamiltonian graph.

This result is quite relevant to the second Hamiltonian problem in claw-free graphs. However, to the best of our knowledge, no one claims to prove the existence of a second Hamiltonian cycle in claw-free graphs with minimum degree at least 3.

In this paper, we take a step toward a solution and prove the existence of the second Hamiltonian cycle in a large subset of the claw-free graphs with minimum degree at least 3. To state the result, we need some definitions.

By a *sun*, we mean a cycle $C = C_n$ with the addition of n vertices called *peaks* each adjacent to a different pair of consecutive vertices of C. The vertices of C are then called the *valleys* of the sun. If some of the peaks are removed, with the restriction that no two consecutive peaks are removed, we call this a *weak sun*. A sun (or more generally a weak sun) is said to have *extra edges* if edges are added between peaks so that the subgraph induced on the peaks is a collection of disjoint cliques.

We can point to the following theorem as our main contribution. In this theorem we exclude weak suns with extra edges and say that all other Hamiltonian claw-free graphs with the minimum degree at least 3 contains a second Hamiltonian cycle. We prove this theorem in Section 2.

Theorem 3. *If G is a Hamiltonian claw-free graph with $\delta(G) \geq 3$, then either contains another Hamiltonian cycle or G is a weak sun with extra edges.*

In the rest of Section 2 we use Theorem 3 and show that every claw-free graph G with $\delta(G) \geq 3$ containing a Hamiltonian cycle C with at least two consecutive vertices each with degree at least 5 contains another Hamiltonian cycle. In addition, in support of Conjecture 2, we show that if G is a 4-regular, Hamiltonian, claw-free graph of order n, then either G contains a second Hamiltonian cycle or $6|n$ and G is a sun with extra edges inducing triangles.

2 Existence of the second Hamiltonian cycle

In this section, first we provide the proof of Theorem 3. After that, we show some immediate applications and results of this theorem in Corollaries 4 and 5.

Proof. (*of Theorem 3*) Given a cycle C with a notion of direction and a vertex $v \in C$, let v^- and v^+ denote the predecessor and successor of v on C respectively. Define a *hop over* v (or simply *hop*) to be the edge of the form v^-v^+. Let C be a Hamiltonian cycle of G and let $e = uv$ be a chord of C that is not a hop. Such a chord must exist since every vertex is incident to at least one chord and if two hops overlap, we can easily find a second Hamiltonian cycle.

First suppose there is no hop over u. Since G is claw-free, one of vu^+ or vu^- must be an edge; without loss of generality, suppose the former. If v^-v^+ is an edge, then the cycle

DOI: 10.20429/tag.2015.020102

$C' = uvu^+ - C - v^-v^+ - C - u$ is a Hamiltonian cycle that is different from C as desired. Thus, there is no hop over v. Since G is claw-free, one of uv^+ or uv^- must be an edge. If $uv^- \in E(G)$, then the cycle $C' = uv^- - C - u^+v - C - u$ is a Hamiltonian cycle that is different from C as desired. Therefore, we may assume $uv^+ \in E(G)$. Similarly, u^+ and v^+ have no hops and the edges u^+v^- and v^+u^- must be present in the graph. This process can be repeated to create a graph like the one pictured in Figure 1. In particular, note that this process creates precisely two vertices, w and x, with hops. Since w must have degree at least 3, there must be at least one more edge out of w, say to a vertex z.

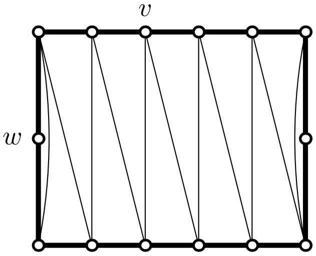

Figure 1: The cycle C with chords

If the edge wz is a hop, then this edge, along with the hop over w, easily allows us to build a second Hamiltonian cycle. Now suppose wz is not a hop. If $z = x$, then a second Hamiltonian cycle is trivial to create using the chords drawn in Figure 1. This means we may assume z is elsewhere on C, say for example $z = u$. There must be no claw at z but also there is no hop over z as observed previously. This means that one of wz^+ or wz^- must be an edge, suppose the former. Then the cycle $C' = wz^+ - C - w^-w^+ - C - zw$ is a Hamiltonian cycle that is different from C as desired.

This implies that there must be a hop over both ends of every chord that is not a hop. Since there can be no hop over an end-vertex of a hop as then a second Hamiltonian cycle would be trivial to find, this means that G must be a weak sun with extra edges. It remains to show that these extra edges indeed induce cliques. Suppose not, that is, suppose there is a vertex u with two incident non-hop chords, say to vertices v and w, such that $vw \notin E(G)$. Note that u, v and w must all have hops over them. Then the claw centered at u with edges to u^+, v and w is induced, for a contradiction, completing the proof of Theorem 3. $\square$

If a Hamiltonian, claw-free graph G has $\delta(G) \geq 3$ and no second Hamiltonian cycle, then Theorem 3 states that G is a weak sun with extra edges. Such a graph contains a *canonical* Hamiltonian cycle C_0 consisting of the valleys in order with the addition of each peak vertex in between its neighboring valleys. Since valley vertices have degree at most 4, if C is a Hamiltonian cycle containing two consecutive vertices of degree at least 5, then C is not canonical so $C \neq C_0$. This means that C is a second Hamiltonian cycle, which yields the following corollary.

Corollary 4. *If G is a claw-free graph with $\delta(G) \geq 3$ containing a Hamiltonian cycle C with at least two consecutive vertices each with degree at least 5, then G contains another Hamiltonian cycle.*

If G is a Hamiltonian, claw-free graph with $\delta(G) \geq 3$ and no second Hamiltonian cycle, then Theorem 3 says that G must be a weak sun with extra edges. If $\delta(G) \geq 4$, the valley vertices must each be adjacent to two peaks, meaning that G must actually be a sun with extra edges. Thus, if G is 4-regular, since the extra edges induce cliques but the peaks already have degree 2, the extra edges must induce triangles. This yields the following corollary.

Corollary 5. *If G is a 4-regular, Hamiltonian, claw-free graph of order n, then either G contains a second Hamiltonian cycle (in support of Conjecture 2) or $6|n$ and G is a sun with extra edges inducing triangles.*

References

[1] H. Bielak. Chromatic properties of Hamiltonian graphs. *Discrete Math.*, 307(11-12):1245–1254, 2007.

[2] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[3] R. C. Entringer and H. Swart. Spanning cycles of nearly cubic graphs. *J. Combin. Theory Ser. B*, 29(3):303–309, 1980.

[4] H. Fleischner. Uniquely Hamiltonian graphs of minimum degree 4. *J. Graph Theory*, 75(2):167–177, 2014.

[5] M. Ghandehari and H. Hatami. A note on independent dominating sets and second hamiltonian cycles. *Manuscript*.

[6] P. Haxell, B. Seamone, and J. Verstraete. Independent dominating sets and Hamiltonian cycles. *J. Graph Theory*, 54(3):233–244, 2007.

[7] J. Petersen. Die Theorie der regulären graphs. *Acta Math.*, 15(1):193–220, 1891.

[8] J. Sheehan. The multiplicity of Hamiltonian circuits in a graph. In *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 477–480. Academia, Prague, 1975.

[9] A. G. Thomason. Hamiltonian cycles and uniquely edge colourable graphs. *Ann. Discrete Math.*, 3:259–268, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[10] C. Thomassen. Independent dominating sets and a second Hamiltonian cycle in regular graphs. *J. Combin. Theory Ser. B*, 72(1):104–109, 1998.

[11] W. T. Tutte. On Hamiltonian circuits. *J. London Math. Soc.*, 21:98–101, 1946.