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An Isomorphism Problem in Z^2

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Abstract

We consider Euclidean distance graphs with vertex set $\mathbb{Q}^2$ or $\mathbb{Z}^2$ and address the possibility or impossibility of finding isomorphisms between such graphs. It is observed that for any distances d_1, d_2 the non-trivial (that is, having non-empty edge set) distance graphs $G(\mathbb{Q}^2, d_1)$ and $G(\mathbb{Q}^2, d_2)$ are isomorphic. Ultimately it is shown that for distinct primes p_1, p_2 the non-trivial distance graphs $G(\mathbb{Z}^2, \sqrt{p_1})$ and $G(\mathbb{Z}^2, \sqrt{p_2})$ are not isomorphic. We conclude with a few additional questions related to this work.

Keywords and phrases: Euclidean distance graph, graph isomorphisms

1 Definitions

For any $\mathbf{X} \subset \mathbb{R}^n$ and $d > 0$, let $G(\mathbf{X}, d)$ be the graph with vertex set $\mathbf{X}$ with any two vertices being adjacent if and only if they are a Euclidean distance d apart. We denote by $\chi(\mathbf{X}, d)$ and $k(\mathbf{X}, d)$ the chromatic number of $G(\mathbf{X}, d)$ and number of components of $G(\mathbf{X}, d)$ respectively. If a given graph $G(\mathbf{X}, d)$ is regular of degree r for some finite value r (which will frequently be the case in this article), we write $\deg(\mathbf{X}, d) = r$.

For graphs G_1 and G_2, we say G_1 is isomorphic to G_2 and write $G_1 \simeq G_2$ if and only if there exists a bijection $f : V(G_1) \to V(G_2)$ such that for any $u, v \in V(G_1)$, u, v are adjacent if and only if $f(u), f(v)$ are adjacent. Additionally, we will employ a bit of non-standard notation in Section 3. For a graph G, vertex $v \in V(G)$, and positive integer l, let $W(G, v, l)$ be the number of distinct closed walks of length l containing v.

2 A Question in $\mathbb{Q}^2$

We begin this work in response to a *Geombinatorics* article of yesteryear – well, fourteen years ago to be exact, but please do not think of us as being late to the party. In 2001, Abrams and Johnson [1] develop the concept of the n^{th} Babai number of a space $\mathbf{X} \subset \mathbb{R}^n$. Denoted as $B_n(\mathbf{X})$, they define this value as

$$B_n(\mathbf{X}) = \max\{\chi(\mathbf{X}, D) \text{ where } D \subset (0, \infty) \text{ and } |D| = n\}.$$

Incidentally, it was later realized in [2] that the word "max" in the definition should be "sup", but "max" is valid whenever $B_n(\mathbf{X})$ is finite, as will be the case here.

Abrams and Johnson conclude by establishing the fact that $B_1(\mathbb{Q}^2) = 2$. In other words, for any distance d realized between points of $\mathbb{Q}^2$, there exists a proper 2-coloring of the graph $G(\mathbb{Q}^2, d)$. However, there is actually more that can be said. We first make note of a well-known characterization due to Euler of integers which can be represented as a sum of two squares.

Lemma 2.1. *A positive integer n may be written as $n = a^2 + b^2$ for $a, b \in \mathbb{Z}$ if and only if in the prime factorization of n, prime factors congruent to* $3 \pmod 4$ *each appear to an even degree.*

Note also that Euler's characterization implies that if a pair of integers x, y are each representable as the sum of two integer squares, then their product xy is also representable as a sum of two integer squares. A relevant extension is the fact that if two non-zero rational numbers q_1, q_2 are each representable as a sum of two rational squares, then both their product and their ratio are representable as a sum of two rational squares. To see this, observe that if $q_1 = {a_1}^2 + {b_1}^2$ and $q_2 = {a_2}^2 + {b_2}^2$ for $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, then $q_1 q_2 = (a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2$ and $\frac{q_1}{q_2} = (\frac{a_1a_2-b_1b_2}{q_2})^2 + (\frac{a_1b_2-b_1a_2}{q_2})^2$. This gives rise to the following theorem.

Theorem 2.2. *For any $q_1, q_2 \in \mathbb{Q}^+$ such that $\sqrt{q_1}$ and $\sqrt{q_2}$ are both distances realized in $\mathbb{Q}^2$, the graphs $G(\mathbb{Q}^2, \sqrt{q_1})$ and $G(\mathbb{Q}^2, \sqrt{q_2})$ are isomorphic.*

Proof. Let $q_1 = rq_2$. By Lemma 2.1 and the above observations, there exist $a, b \in \mathbb{Q}$ such that $a^2 + b^2 = r$. Define a transformation $f : \mathbb{Q}^2 \to \mathbb{Q}^2$ where for each $(x, y) \in \mathbb{Q}^2$, $f(x, y) = (ax + by, bx - ay)$. It is easily seen that f scales distance by a factor of $\sqrt{r}$. That is, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}^2$, $\|f(x_1, y_1) - f(x_2, y_2)\| = \sqrt{r}\|(x_1, y_1) - (x_2, y_2)\|$. Furthermore, this transformation is a bijection as the matrix $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ is invertible. Thus $G(\mathbb{Q}^2, \sqrt{q_1}) \simeq G(\mathbb{Q}^2, \sqrt{q_2})$. □

So of course all non-trivial Euclidean distance graphs with vertex set $\mathbb{Q}^2$ have the same chromatic number. They are all just the same graph. This observation is not noteworthy in itself. It is basically just a classical argument presented in a new setting. However, we are now wondering what happens when we ask a similar question about Euclidean distance graphs with vertex set $\mathbb{Z}^2$. In other words, for distinct positive integers z_1, z_2 with $\sqrt{z_1}, \sqrt{z_2}$ each being realized as a distance in $\mathbb{Z}^2$, is it possible that $G(\mathbb{Z}^2, \sqrt{z_1}) \simeq G(\mathbb{Z}^2, \sqrt{z_2})$?

3 A Question in $\mathbb{Z}^2$

Given $G(\mathbb{Z}^2, \sqrt{z_1})$ and $G(\mathbb{Z}^2, \sqrt{z_2})$, there are two immediate conditions which would result in the graphs being non-isomorphic. If $k(\mathbb{Z}^2, \sqrt{z_1}) \neq k(\mathbb{Z}^2, \sqrt{z_2})$ or if $\deg(\mathbb{Z}^2, \sqrt{z_1}) \neq \deg(\mathbb{Z}^2, \sqrt{z_2})$, then certainly we would have that $G(\mathbb{Z}^2, \sqrt{z_1}) \not\simeq G(\mathbb{Z}^2, \sqrt{z_2})$. These issues, however, can be addressed with elementary arguments along with an appeal to classical number theory.

To begin, note that for any graph $G(\mathbb{Z}^2, \sqrt{z})$, $\deg(\mathbb{Z}^2, \sqrt{z})$ is simply the number of different ordered signed representations of z as a sum of two integer squares. This number is well-known (see [3]), and extends Lemma 2.1 from the previous section.

Lemma 3.1. *Let $z \in \mathbb{Z}^+$ be written as $z = 2^\alpha {p_1}^{\beta_1} \cdots {p_m}^{\beta_m} {q_1}^{2\gamma_1} \cdots {q_n}^{2\gamma_n}$ with each p_i being a distinct prime congruent to* 1 (mod 4) *and each q_j being a distinct prime congruent to* 3 (mod 4). *Then the number of different ordered signed representations of z as a sum of two integer squares is $4(\beta_1 + 1)(\beta_2 + 1) \cdots (\beta_m + 1)$.*

As it turns out, Lemma 3.1 not only gives $\deg(\mathbb{Z}^2, \sqrt{z})$. It is particularly useful in determining $k(\mathbb{Z}^2, \sqrt{z})$ as well.

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Theorem 3.2. *Let $z \in \mathbb{Z}^+$ such that the prime factorization of z consists solely of factors congruent to* $1 \pmod 4$. *Then $G(\mathbb{Z}^2, \sqrt{z})$ is connected.*

Proof. We first show that the vector $\langle 0, 1 \rangle$ can be realized as a sum of $\mathbb{Z}^2$ vectors of length $\sqrt{z}$. By Lemma 3.1, there exist integers a, b such that $a^2 + b^2 = z$ and $\gcd(a, b) = 1$. As z is odd, we have that exactly one of a, b is even. So assume a is even and for the sake of clarity, assume a and b are non-negative. As $\gcd(a, b) = 1$, let s, t be non-negative integers such that $sa - tb = -1$. Then $\langle a, b \rangle + \frac{a}{2}[s\langle a, b \rangle + s\langle a, -b \rangle + t\langle -b, a \rangle + t\langle -b, -a \rangle] + \frac{b-1}{2}[s\langle b, a \rangle + s\langle -b, a \rangle + t\langle a, -b \rangle + t\langle -a, -b \rangle] = \langle 0, 1 \rangle$. The fact that $\langle 0, 1 \rangle$ can be expressed as a sum of $\mathbb{Z}^2$ vectors of length $\sqrt{z}$ implies that $\langle 1, 0 \rangle, \langle -1, 0 \rangle$, and $\langle 0, -1 \rangle$ can be as well. As any vector in $\mathbb{Z}^2$ can be expressed as a combination of the vectors $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle -1, 0 \rangle$, and $\langle 0, -1 \rangle$, we have established that $G(\mathbb{Z}^2, \sqrt{z})$ is connected. □

Theorem 3.3. *Let $z \in \mathbb{Z}^+$ where the prime factorization of z consists solely of factors congruent to* $1 \pmod 4$. *Then $k(\mathbb{Z}^2, \sqrt{2z}) = 2$.*

Proof. By Lemma 3.1, there exist $a, b \in \mathbb{Z}^+$ with $a^2 + b^2 = 2z$ and $\gcd(a, b) = 1$. However, note here that a, b are both odd. It follows that points $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2$ are in the same component of $G(\mathbb{Z}^2, \sqrt{2z})$ only if $x_1 + y_1 \equiv x_2 + y_2 \pmod 2$. Thus $k(\mathbb{Z}^2, \sqrt{2z}) \geq 2$. We now mimic the proof of Theorem 3.2, except this time we create the vector $\langle 2, 0 \rangle$ as a sum of $\mathbb{Z}^2$ vectors of length $\sqrt{2z}$. Let $s, t \in \mathbb{Z}^+$ such that $sa - tb = 1$. Then $s\langle a, b \rangle + s\langle a, -b \rangle + t\langle -b, a \rangle + t\langle -b, -a \rangle = \langle 2, 0 \rangle$. By similar arguments, $\langle 0, 2 \rangle, \langle -2, 0 \rangle$, and $\langle 0, -2 \rangle$ can be created as a sum of the desired vectors as well. Furthermore, we may then add some number of the vectors $\langle -2, 0 \rangle$ and $\langle 0, -2 \rangle$ to $\langle a, b \rangle$ to create $\langle 1, 1 \rangle$. Now observing that any $\mathbb{Z}^2$ vector can be obtained by beginning with either $\langle 0, 0 \rangle$ or $\langle 1, 0 \rangle$ and adding to that vector some number of the vectors $\langle 2, 0 \rangle, \langle 0, 2 \rangle, \langle -2, 0 \rangle, \langle 0, -2 \rangle$, or $\langle 1, 1 \rangle$, we have that $k(\mathbb{Z}^2, \sqrt{2z}) = 2$. □

Regarding Lemma 3.1, note that for any integers a, b and prime $q \equiv 3 \pmod 4$, $q^2 | (a^2 + b^2)$ implies that $q|a$ and $q|b$. Thus in the graph $G(\mathbb{Z}^2, \sqrt{zq^2})$ where z is any positive integer, vertices (x_1, y_1) and (x_2, y_2) are in the same component only if $x_1 \equiv x_2 \pmod q$ and $y_1 \equiv y_2 \pmod q$. Extending this fact in conjunction with the previous two theorems, we obtain the following result.

Theorem 3.4. *Let $z = 2^\alpha {p_1}^{\beta_1} \cdots {p_m}^{\beta_m} {q_1}^{2\gamma_1} \cdots {q_n}^{2\gamma_n}$ with each p_i being a distinct prime congruent to* $1 \pmod 4$ *and each q_j being a distinct prime congruent to* $3 \pmod 4$. *Then $k(\mathbb{Z}^2, \sqrt{z}) = 2^\alpha {q_1}^{2\gamma_1} \cdots {q_n}^{2\gamma_n}$.*

In light of the preceding work, it appears that a thorough analysis of the question posed at the end of Section 2 can be reached by only considering graphs of the form $G(\mathbb{Z}^2, \sqrt{z})$ where the prime factorization of z consists solely of factors congruent to $1 \pmod 4$. We begin this analysis below, first making note of a graph invariant that we will use along the way.

Lemma 3.5. *Let G, H be graphs and let $f : V(G) \to V(H)$ be an isomorphism. Then for any $v \in V(G)$ and $l \in \mathbb{Z}^+$, $W(G, v, l) = W(H, f(v), l)$.*

Proof. We need only note that there is a bijection ϕ between the set of all closed walks in G containing v and the set of all closed walks in H containing $f(v)$. For any closed walk $(v, v_1, ..., v_{l-1}, v)$ in G, just let $\phi(v, v_1, ..., v_{l-1}, v) = (f(v), f(v_1), ..., f(v_{l-1}), f(v))$. □

Theorem 3.6. *Let p, q be distinct primes congruent to* $1 \pmod 4$. *Then* $G(\mathbb{Z}^2, \sqrt{p}) \not\cong G(\mathbb{Z}^2, \sqrt{q})$.

Proof. For the sake of brevity, let $G_1 = G(\mathbb{Z}^2, \sqrt{p})$ and let $G_2 = G(\mathbb{Z}^2, \sqrt{q})$. For any $u_1, u_2 \in \mathbb{Z}^2$ and any $l \in \mathbb{Z}^+$, we have that $W(G_1, u_1, l) = W(G_1, u_2, l)$. Similarly, for any $v_1, v_2 \in \mathbb{Z}^2$ and any $l \in \mathbb{Z}^+$, we have that $W(G_2, v_1, l) = W(G_2, v_2, l)$. So to calculate $W(G_1, u, l)$ for any vertex u we need only determine the number of distinct ordered collections of l vectors, each having integer entries and length $\sqrt{p}$, whose sum is the zero vector. We calculate $W(G_2, v, l)$ in the same fashion except requiring that each vector has length $\sqrt{q}$. To establish that $G_1 \not\cong G_2$, it suffices to show that for some $l \in \mathbb{Z}^+$, $W(G_1, u, l) \neq W(G_2, v, l)$.

In the interest of economy of words, we will refer to an ordered collection of vectors that sum to the zero vector as just being a "collection". Furthermore, we will classify each collection as being trivial or non-trivial. A trivial collection is one where for each integer z, the number of times z appears as an x-component entry in the collection is equal to the number of times $-z$ appears as an x-component entry in the collection and the number of times z appears as an y-component entry in the collection is equal to the number of times $-z$ appears as an y-component entry in the collection. A non-trivial collection is any that does not have this property.

Let $a, b, c, d \in \mathbb{Z}^+$ such that $a^2 + b^2 = p$ and $c^2 + d^2 = q$. By Lemma 3.1, the vectors $\langle a, b\rangle$ and $\langle c, d\rangle$, along with those formed by permuting entries or replacing an entry with its negative, are the only vectors of length $\sqrt{p}$ and $\sqrt{q}$ respectively. It follows that for any $l \in \mathbb{Z}^+$, the number of distinct trivial collections of l vectors of length $\sqrt{p}$ is equal to the number of distinct trivial collections of l vectors of length $\sqrt{q}$.

We now determine the minimum value l such that a non-trivial collection of l vectors of length $\sqrt{p}$ actually exists. First note that if such a non-trivial collection is minimum, then either the set of all x-entries in the collection cannot contain both a and $-a$ or b and $-b$ or the set of all y-entries in the collection cannot contain both a and $-a$ or b and $-b$. Without loss of generality, we may assume a minimum collection contains only a and $-b$ as x-entries of the vectors in that collection. Therefore, a minimum collection must be of length $s + t$ where $s, t \in \mathbb{Z}^+$ and $sa - tb = 0$. As $\gcd(a, b) = 1$, we have that $(s + t)$ is a multiple of $(a + b)$. However, $s + t \neq a + b$ as the y-component entries of this collection would then consist of b copies of $\pm b$ and a copies of $\pm a$. As exactly one of a, b is even, it would then be impossible for the y-entry of the sum of this collection of vectors to be zero. Now noting that $a\langle -b, a\rangle + a\langle -b, -a\rangle + b\langle a, b\rangle + b\langle a, -b\rangle = \langle 0, 0\rangle$, we have established that a minimum non-trivial collection of l vectors of length $\sqrt{p}$ has $l = 2(a + b)$. By a similar argument, a minimum non-trivial collection of l vectors of length $\sqrt{q}$ has $l = 2(c+d)$. So if $a+b \neq c+d$, we have that $G(\mathbb{Z}^2, \sqrt{p}) \not\cong G(\mathbb{Z}^2, \sqrt{q})$ and we're done.

Now assume that $2(a + b) = 2(c + d) = l$. Consider a non-trivial collection of l vectors of length $\sqrt{p}$ whose x-component entries consist solely of a and $-b$. To construct such a collection, we have $\binom{2a+2b}{2b}$ choices of how to distribute the $2b$ copies of a among the ordered x-entries of the vectors. Once we make such an assignment of values to the x-entries, we then have $\binom{2a}{a}$ choices of how to distribute the a copies each of the terms a and $-a$ among the ordered y-entries of the vectors. Similarly, we have $\binom{2b}{b}$ choices of how to distribute the b copies each of the terms b and $-b$ among the ordered y-entries of the vectors. Combining these facts, and observing that we may create additional non-trivial collections of l vectors

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of length $\sqrt{p}$ by simultaneously exchanging the x- and y-entries of each of the l vectors or by simultaneously replacing each copy of a with $-a$ and each copy of $-b$ with b in each vector of the collection, we obtain the fact that the number of distinct non-trivial collections of l vectors of length $\sqrt{p}$ is given as $4\binom{2a+2b}{2b}\binom{2a}{a}\binom{2b}{b}$, which may be simplified to $4\frac{l!}{(a!b!)^2}$. Moreover, the number of distinct non-trivial collections of l vectors of length $\sqrt{q}$ is equal to $4\frac{l!}{(c!d!)^2}$. Let $n = a + b = c + d$ and observe that a, b, c, d must be distinct. Without loss of generality, assume $a < c < d < b$ and note that this implies $a \leq \frac{n-1}{2} - 1$. Define a function $f(x) = x!(n-x)!$ and consider the behavior of $f(x)$ on the domain $\{1, 2, ..., \frac{n-1}{2} - 1\}$. We have that $\frac{f(x+1)}{f(x)} = \frac{(x+1)!(n-(x+1))!}{x!(n-x)!} = \frac{x+1}{n-x}$ and as $x \leq \frac{n-1}{2} - 1$, it follows that $x + 1 \leq \frac{n-1}{2}$ and $n - x \geq n - (\frac{n-1}{2} - 1) = \frac{n+3}{2}$. Thus $\frac{x+1}{n-x} \leq \frac{n-1}{n+3} < 1$. This means that $f(x)$ is decreasing on $\{1, 2, ..., \frac{n-1}{2}\}$ which in turn implies that $a!b! \neq c!d!$. By extension, $W(G_1, u, l) \neq W(G_2, v, l)$ and we have proven that $G(\mathbb{Z}^2, \sqrt{p}) \not\simeq G(\mathbb{Z}^2, \sqrt{q})$. □

4 Further Work

It is our guess that there do not exist distinct $z_1, z_2 \in \mathbb{Z}^+$ such that the non-trivial distance graphs $G(\mathbb{Z}^2, \sqrt{z_1})$ and $G(\mathbb{Z}^2, \sqrt{z_2})$ are isomorphic. In light of the work done in the previous section, we may completely resolve this matter by considering pairs of graphs of the form $G(\mathbb{Z}^2, \sqrt{z_1}), G(\mathbb{Z}^2, \sqrt{z_2})$ where z_1, z_2 are composite, have prime factorizations consisting solely of factors congruent to $1 \pmod 4$, and if $\alpha_1, ..., \alpha_m$ and $\beta_1, ..., \beta_n$ are the multiplicities of the prime factors of z_1 and z_2 respectively, then $(\alpha_1 + 1)\cdots(\alpha_m + 1) = (\beta_1 + 1)\cdots(\beta_n + 1)$. So at least that narrows down our focus. However, there is another possibly interesting direction we can take this question. A taste of this can be seen in the following example.

Consider the graphs $G(\mathbb{Z}^2, \sqrt{5})$ and $G(\mathbb{Z}^2, \sqrt{p})$ where p is any prime greater than 5. By Theorem 3.6, we know $G(\mathbb{Z}^2, \sqrt{5}) \not\simeq G(\mathbb{Z}^2, \sqrt{p})$. We are now wondering if this can be shown directly by constructing a graph H that is a subgraph of one of these graphs but not of the other. Well, it just so happens that it can.

Let H be the graph in Figure 1, where it is presented as a subgraph of $G(\mathbb{Z}^2, \sqrt{5})$.

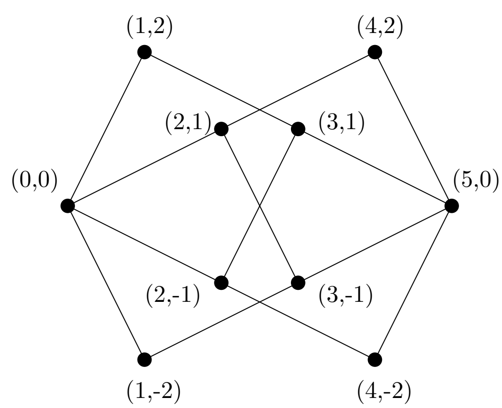

Figure 1

Theorem 4.1. *For any prime $p > 5$, H is not a subgraph of $G(\mathbb{Z}^2, \sqrt{p})$.*

Proof. Suppose H does appear as a subgraph of $G(\mathbb{Z}^2, \sqrt{p})$. Label the vectors corresponding to the edges of H as in Figure 2.

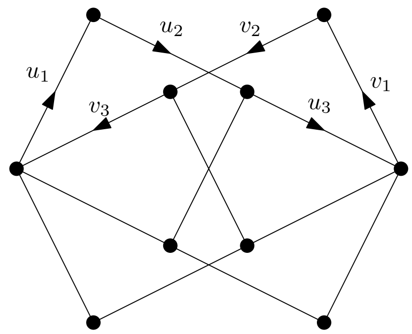

Figure 2

Note that we are depicting H with the same visual representation as it has when appearing as a subgraph of $G(\mathbb{Z}^2, \sqrt{5})$. This certainly does not have to be the case, but we feel it aids in grasping the argument that follows. Note also that once we have labeled $u_1, u_2, u_3, v_1, v_2, v_3$ as in Figure 2, our hand is forced for labeling the remaining edges with their corresponding vectors. This is done in Figure 3.

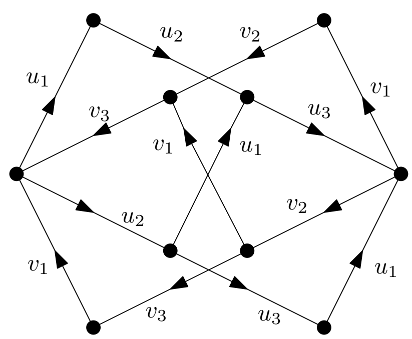

Figure 3

We now observe that $u_1 + u_2 + u_3 + v_1 + v_2 + v_3 = \mathbf{0}$, but that none of the following sums of two vectors equal the zero vector: $u_1 + u_2, u_1 + u_3, u_1 + v_1, u_1 + v_2, u_1 + v_3, u_2 + u_3, u_2 + v_1, u_2 + v_3, u_3 + v_1, u_3 + v_2, v_1 + v_2, v_1 + v_3, v_2 + v_3$. Furthermore, it must be that $u_1 \neq u_2, u_1 \neq u_3, v_1 \neq v_2$, and $v_1 \neq v_3$. Let $p = a^2+b^2$ for $a, b \in \mathbb{Z}^+$ and note that $a+b > 3$. In regards to the work done in the proof of Theorem 3.6, we have that $T = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ is a trivial collection of six vectors of length $\sqrt{p}$ that sum to the zero vector. We need only consider the following two cases.

Case 1 Suppose the x-entries of the six vectors of T consist of two copies of a, two copies of $-a$, and one copy each of b and $-b$.

By the above observations, for any vector $v \in T$, $u_1 + v \neq \mathbf{0}$ and $v_1 + v \neq \mathbf{0}$. As the two vectors with x-entries of b or $-b$ must be additive inverses, it follows that neither of those vectors can be equal to u_1 or v_1. Also in accordance with the previous observations, it follows that one of the two vectors with x-entries of b or $-b$ is equal to u_i for $i \in \{2,3\}$ if and only if the other of the two vectors is equal to v_i.

Assuming the two vectors with x-entries of b or $-b$ are equal to u_2 and v_2, since $u_1 \neq u_3$ and $u_1 + u_3 \neq \mathbf{0}$, we have that u_1 and u_3 agree on exactly one of their two component entries. This implies that either $u_1 + v_1 = \mathbf{0}$ or $u_3 + v_1 = \mathbf{0}$, contradicting our previous observations.

Assuming the two vectors with x-entries of b or $-b$ are equal to u_3 and v_3, since $u_1 \neq u_2$ and $u_1 + u_2 \neq \mathbf{0}$, we have that u_1 and u_2 agree on exactly one of their two component entries. This implies that either $u_1 + v_1 = \mathbf{0}$ or $u_2 + v_1 = \mathbf{0}$, again contradicting our previous observations.

Case 2 Suppose the x-entries of the six vectors of T consist of three copies of a and three copies of $-a$.

We have that $u_1 + v_1 \neq \mathbf{0}$. If $u_1 \neq v_1$, we would then have that each of u_2, u_3, v_2, v_3 would be an additive inverse for one of u_1, v_1, contradicting our previous observations. Assuming that $u_1 = v_1$ and again noting that none of u_2, u_3, v_2, v_3 are additive inverses of u_1, v_1 and noting that $u_1 \notin \{u_2, u_3\}$ and $v_1 \notin \{v_2, v_3\}$, it follows that the set $\{u_2, u_3, v_2, v_3\}$ contains at most two distinct vectors, and if it contains two, then those two vectors are additive inverses of each other. As $u_2 + u_3 \neq \mathbf{0}$ and $u_2 + v_3 \neq \mathbf{0}$, it follows that $u_2 = u_3 = v_3$. As $v_2 + v_3 \neq \mathbf{0}$ and $v_2 + u_3 \neq \mathbf{0}$, it follows that $v_2 = v_3 = u_3$. Thus $u_2 = u_3 = v_2 = v_3$, contradicting the fact that the x-entries of T sum to 0. □

Admittedly, the proof of Theorem 4.1 is a bit tedious and awkward. However, it does serve as a solution to a specific instance of the following general question, which we feel is of interest and offers an avenue for future work.

Question 1 Given non-isomorphic graphs $G(\mathbb{Z}^2, d_1)$ and $G(\mathbb{Z}^2, d_2)$, if possible, construct a finite graph H which appears as a subgraph of exactly one of the two given graphs.

Better yet, answer the following.

Question 2 Given non-isomorphic graphs $G(\mathbb{Z}^2, d_1)$ and $G(\mathbb{Z}^2, d_2)$, construct a graph H as stipulated by Question 1 where either $|V(H)|$ or $|E(H)|$ is minimum.

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