2015

Bounds for the Zero Forcing Number of Graphs with Large Girth

Randy Davila
Rice University, rrd6@rice.edu

Franklin Kenter
Rice University, fhk2@rice.edu

Bounds for the Zero Forcing Number of Graphs with Large Girth

Abstract

The zero-forcing number, $Z(G)$ is an upper bound for the maximum nullity of all symmetric matrices with a sparsity pattern described by the graph. A simple lower bound is $\delta \leq Z(G)$ where δ is the minimum degree. An improvement of this bound is provided in the case that G has girth of at least 5. In particular, it is shown that $2\delta - 2 \leq Z(G)$ for graphs with girth of at least 5; this can be further improved when G has a small cut set. Lastly, a conjecture is made regarding a lower bound for $Z(G)$ as a function of the girth and δ; this conjecture is proved in a few cases and numerical evidence is provided.

1 Introduction

The problem of determining the minimum rank of a graph seeks the minimum rank over all symmetric matrices whose sparsity pattern is determined by the graph. The zero forcing process, and the associated zero forcing number, were introduced by [1] and [6] in order to bound the minimum rank of a graph (and hence the maximum nullity). Further, the zero forcing process and its variants have expanded applications in power network monitoring [13, 17], quantum physics [6], and logic [7]. Since then, the zero forcing number has gained much attention in graph theory and has been related to many graph theoretic parameters including minimum degree, d'arboresense, treewidth, pathwidth, among others [3]. We give a detailed description of the zero forcing process in the next section.

In general, the study of the zero forcing number is challenging for many reasons. First, it is difficult to compute exactly, as it is NP-hard [9]. Further, many of the known bounds leave a wide gap for graphs in general. For example, given the minimum and maximum degree of a graph, δ and Δ respectively, the zero forcing number on a graph with n vertices can be as low as δ [3] , and as high as $\frac{n\Delta}{\Delta+1}$ [2].

To date, the only result concerning minimum rank or zero forcing and triangle-free graphs is given by Deaett [10] where it is shown that the *semidefinite minimum rank* of a triangle-free graph is bounded below by half the number of vertices. Our main contribution is the improvement of the lower bound mentioned above in the case of triangle free graphs and graphs with larger girth.

This paper is organized as follows: In Section 2 we give the basic definitions and background. Next, we provide our main results with proofs in Section 3. Finally, in Section 4, we conjecture improvements to our main results and provide computational evidence.

2 Preliminaries

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. We will use standard notation: $n = |V|$, $m = |E|$, minimum vertex degree $\delta(G) = \delta$, and maximum vertex degree $\Delta(G) = \Delta$. The *girth* of a graph G, denoted g, is the length of its shortest cycle. If G is a tree, by convention, we say that $g = \infty$. When $g > 3$, we call a graph *triangle-free*. The *open neighborhood* of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v, excluding v itself. The *closed neighborhood*, $N[v]$, includes v. In addition, given $S \subset V$, let $G[S]$ denote the graph induced by S.

We now define the *zero forcing process* as first described by [1]: Let $S = S_0$ be an initial set of "colored" vertices. All other vertices are said to be "uncolored". At each time step, all colored vertices with exactly one uncolored neighbor will change, or *force*, their uncolored neighbor to be colored (note that two or more vertices may force the same vertex to become colored). A set $S \subset V$ of initially color vertices is called a *zero forcing set*, if by iteratively applying the zero forcing process, all of V becomes colored. The *zero forcing number* of a graph G, denoted by $Z(G)$, is the cardinality of a smallest zero forcing set of G, and has been shown to be an NP-hard invariant [10, 16].

In our analysis, we will pay close attention as to when each vertex becomes colored and forces in the iterative zero forcing process. We say the process is at *time t* if the color-change rule has been applied exactly t times. The set of colored vertices at time t is denoted S_t. For convenience, we will call a vertex *active* at time t if it will force at time $t + 1$.

A classical lower bound for the zero forcing number is the following:

Proposition 2.1 (see, for example, [1, 3])**.** *For any graph G,*

$$Z(G) \geq \delta.$$

In a similar manner, Amos, Caro, Davila, and Pepper recently provided an upper bound for the zero-forcing number:

Proposition 2.2 (Amos-Caro-Davila-Pepper [2])**.** *For any graph G with $\delta \geq 1$,*

$$Z(G) \leq \frac{n\Delta}{\Delta + 1}.$$

Further, it is clear that $Z(G) = n-1$ if and only if $G = K_n$. This result leads to the following proposition.

Proposition 2.3 (see, for example [15])**.** *Let G be a graph with non-empty components and order n. Then,*

$$Z(G) \leq n - 2,$$

whenever $G \neq K_n$.

One of the primary motivations of zero forcing is the study of minimum rank of a graph. Given a field $\mathcal{F}$, the minimum rank of a graph over $\mathcal{F}$, denoted $mr_{\mathcal{F}}(G)$, is the minimum rank over all symmetric $n \times n$ matrices $\mathbf{M}$ with entries from $\mathcal{F}$ such that $\mathbf{M}_{ij} = 0$ whenever $i \neq j$ and $\{i, j\}$ is not an edge and $\mathbf{M}_{ij} \neq 0$ when $\{i, j\}$ is an edge; the diagonal entries may take any value. The important connection between the minimum rank of graph and zero forcing is the following:

Proposition 2.4 (AIM-Group [1])**.** *For any graph G and any field $\mathcal{F}$,*

$$Z(G) \geq n - mr_{\mathcal{F}}(G).$$

3 Main Results

In this section, we describe our main results. The main idea of this paper is that Proposition 2.1 can be improved under mild requirements.

Lemma 3.1. *Let G be a triangle-free graph with $\delta \geq 3$. Let S be a minimum zero forcing set of G. Let $v \in S$ force w at time $t = 1$, then w has at least one neighbor not in S.*

Proof. Let G be a graph with $\delta \geq 3$, and let S_0 be a minimum zero forcing set of G such that v forces w at time $t = 1$. By way of contradiction, assume that $N(w) \subset S_0$. Since $\delta \geq 3$, we know there exists $z \in N(w) \setminus \{v\}$. Since G is triangle-free, we know $z \notin N(v)$. Starting with a uncolored copy of G, define an initial set of colored vertices $S_0' = S_0 \setminus \{z\}$. Since v is not adjacent to z, v forces w at time $t = 1$ under the new coloring S_0'. Since we have assumed that $N(w)$ is initially colored in S_0, we know that $N(w) \setminus \{z\}$ is colored in S_0'. It follows that at time $t = 2$, w will have $d(w) - 1$ neighbors colored, and hence will force z. So we have shown that $S_0 \subset S_2'$. Since S_0 was a zero forcing set of G, S_0' must also be a zero forcing set of G, contradicting the minimality of S_0. $\qquad\square$

Theorem 3.2. *Let G be a triangle-free graph with minimum degree $\delta \geq 3$. Then,*

$$\delta + 1 \leq Z(G).$$

Proof. Suppose G is triangle-free, and let S be a zero forcing set realizing $Z(G)$. Since G is not a complete graph, nor an empty graph, by Proposition 2.3, we know at least two initially uncolored vertices exists and hence at least two vertices must force. Let v and w denote these two forcing vertices. Note that v and w may force at separate times. Without loss of generality, suppose v forces v', at time $t = 1$. Hence, $N[v] \setminus \{v'\} \subseteq S$. It suffices to show that there is an initially colored vertex not in $N[v] \setminus \{v'\}$ at time $t = 1$.

Since w is assumed to force eventually, we have the following cases:

1. w forces at time $t = 1$.

2. w forces at time $t \geq 2$.

Case 1. Suppose w forces w', at time $t = 1$. If w is not in the neighborhood of v, then since w was initially colored, we are done. Hence, suppose w is in the neighborhood of v. Then because G is triangle-free, w cannot be adjacent to any neighbors of v, otherwise G would have a triangle. Furthermore, by assumption $\delta \geq 3$, so w must have at least one neighbor other than v and w', z (say). Since w forces w' at time $t = 1$, z must be colored at time $t = 0$. Since w is adjacent to v, z cannot be a neighbor of v, otherwise G would contain a triangle. Therefore, z is initially colored outside $N[v] \setminus \{w\}$, and we are done. Altogether if there are two vertices v and w that force at time $t = 1$, the theorem holds. For for the remaining case we may assume that v is the unique forcing vertex at time $t = 1$.

Case 2. Suppose that w forces at some time $t \geq 2$. Since forcing steps occur at each time step, there must be a vertex that forces at time $t = 2$; without loss of generality, assume that it is w. Since w became active at time $t = 1$, and since v is assumed to be the unique forcing vertex at time $t = 1$, it must be the case that either v forced w or a neighbor of w. If v forces w, then we are done by Lemma 3.1. Otherwise v forced a neighbor of w, and therefore w cannot be in the neighborhood of v as G would contain a triangle. □

The following lemma is a useful tool for finding large zero forcing sets.

Lemma 3.3. *Let S be a set of colored vertices of G, and let $B \subseteq S$ be a set of vertices whose entire neighborhood is colored. Then S is a zero forcing set of G if and only if $S \setminus B$ is a zero forcing set of $G[V \setminus B]$.*

Proof. Necessity: Suppose that $S \setminus B$ is a zero forcing set of G. It follows that coloring S along with any vertices outside of S must also be a zero forcing set. Hence, $S \cup B$ is a zero forcing set of G.

Sufficiency: Suppose that $S \subseteq V$ is a zero forcing set of G. Let $B \subseteq S$ be a set of colored vertices with only colored neighbors. Note that every vertex of B is inactive. Furthermore, any currently active vertex will be active in the graph $V \setminus B$, since no vertex in B has any white neighbors, and hence any active vertex in S will have exactly one white neighbor in $V \setminus B$. Forcing chains will hence proceed as if they were in G, and must force the rest of the graph. □

The following theorem improves on Proposition 2.2 for some families of graphs, such as cycles, but in general is non-comparable.

Theorem 3.4. *Let G be a graph with girth $g \geq 3$. Then,*

$$Z(G) \leq n - g + 2.$$

Proof. Let G be a graph with girth g. Let $C \subseteq V$, be set of vertices realizing the girth of G that induce a cycle. Clearly, $V \setminus C$ together with any two neighbors in C form a zero forcing set of G. □

Lemma 3.5. *Let G be a triangle-free graph with minimum degree $\delta \geq 2$. If there exists a minimum zero forcing set which requires at least two neighbors to force at time $t = 1$, then*

$$2\delta - 2 \leq Z(G).$$

Proof. Since $\delta = 2$ satisfies the Lemma 3.5 trivially, assume G is a triangle-free graph with $\delta \geq 3$ and with a zero forcing set S which requires at least two neighbors to force at time $t = 1$. Let v and w be elements of S which are neighbors and both force at time $t = 1$. Since G is triangle-free, v and w cannot share any neighbors. Hence the colored neighbors of v and w contribute together at least $2(\delta - 1)$ members of the zero forcing set S. □

For graphs with girth greater than 4, we are able to show that the minimum degree lower bound can be improved by a factor of almost 2.

Theorem 3.6. *Let G be a graph with girth $g \geq 5$, and minimum degree $\delta \geq 2$. Then,*

$$2\delta - 2 \leq Z(G).$$

Proof. Notice that when $\delta = 2$, the theorem holds. So we consider graphs with $\delta \geq 3$.

Let G be a graph with girth $g \geq 5$, and minimum degree $\delta \geq 3$. Let S be a zero forcing set realizing $Z(G)$. Since G is not a complete graph, nor an empty graph, we know at least two forcing vertices exist, v and w (say). Note that v and w may force at separate times. Without loss of generality, suppose v forces v' (say), at time $t = 1$. Hence, $N[v] \setminus \{v'\} \subseteq S$.

Since w is assumed to force eventually, we have the following cases:

1. w forces at time $t = 1$.

2. w forces at time $t \geq 2$.

Case 1. Suppose w forces w' (say), at time $t = 1$. Note w is either in the neighborhood of v, or it isn't. If w is not in the neighborhood of v, then because w and all neighbors of w other then w' were colored at time $t = 0$, we have

$$|S| \geq \delta + \delta - k,$$

Where k is the (possibly zero) number of vertices in

$$\big(N[v] \setminus \{v'\}\big) \cap \big(N[w] \setminus \{w'\}\big).$$

Because $g \geq 5$, w cannot be adjacent to more than one neighbor of v, since otherwise v, w, and the two shared neighbors would induce a 4-cycle. Hence,

$$
\begin{aligned}
|S| &\geq & \delta + \delta - 1 \\
&= & 2\delta - 1 \\
&> & 2\delta - 2,
\end{aligned}
$$

and the theorem holds.

Next suppose w is in the neighborhood of v. Then, because G is triangle-free with $\delta \geq 3$, we recall lemma 3.5, and get,

$$|S| \geq 2\delta - 2.$$

If w forces at time $t = 1$, the theorem holds. Since w was arbitrary, if any vertex other than v forces at time $t = 1$, then the theorem holds. Hence, for Case 2 we may assume that v is the unique forcing vertex at time $t = 1$.

Case 2. Suppose that w forces at some time $t \geq 2$. Since forcing steps occur at each time step, there must be a vertex that forces at time $t = 2$, without loss of generality we take w to be this vertex. Since w became active at time $t = 1$, and since v is assumed to be the unique forcing vertex at time $t = 1$, it must be the case that v forced a neighbor of w, i.e., v' is a neighbor of w. Because v forced a neighbor of w, w cannot be in the neighborhood of v, since otherwise v,v', and w would induce a triangle. It follows that w was initially colored. Furthermore, since w is a forcing vertex at time $t = 2$, it must have at least $\delta - 1$ neighbors that are colored at time $t = 1$. Since v was the unique forcing vertex at time $t = 1$, the neighbors of w colored at time $t = 1$, must have been initially colored at time $t = 0$. Since $g \geq 5$, v and w cannot share neighbors, since otherwise the neighbors v and w share, along with v, v', and w, would induce a 4-cycle. Hence,

$$
\begin{aligned}
|S| &\geq & \big|N[v] \setminus \{v'\} \cup N[w] \setminus \{w'\}\big| \\
&= & \big|N[v] \setminus \{v'\}\big| + \big|N[w] \setminus \{w'\}\big| \\
&\geq & \delta + \delta \\
&= & 2\delta \\
&> & 2\delta - 2,
\end{aligned}
$$

and the theorem holds. $\qquad\square$

Before moving to the proofs of Theorems 3.8 and 3.9, we make note of a theorem attributed to Edholm Hogben Hyunh LaGrange and Row:

Theorem 3.7 (Edholm-Hogben-Hyunh-LaGrange-Row [11])**.** *Let G be a graph on $n \geq 2$ vertices, then:*

1. *For $v \in V(G)$, $Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$.*

2. *For $e \in E(G)$, $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$.*

First we consider graphs with a cut vertex, such that after removal of such a vertex, the resulting graph has at least one component with girth at least 5. We improve on Proposition 2.1 by close to a factor of 3 for such graphs.

Theorem 3.8. *Let G be a graph with $\delta \geq 3$ and a cut vertex v such that $G - v$ has a component with girth at least 5. Then,*

$$3\delta - 6 \leq Z(G).$$

Proof. Let G be a graph with minimum degree $\delta \geq 3$ and a cut vertex v such that $G - v$ has a component with girth at least 5. Let $G - v$ have components H_1 and H_2. Without loss of generality suppose H_1 has girth at least 5. Recall $Z(G - v) \leq Z(G) + 1$ by Theorem 3.7. Since zero forcing is additive with respect to disjoint components we have

$$Z(H_1) + Z(H_2) \leq Z(G) + 1.$$

Further, note that $\delta(H_1) \geq \delta - 1$ and $\delta(H_2) \geq \delta - 1$, since deleting v from G at most reduced the degree of the components by 1. By Theorem 3.6, we know that

$$2(\delta - 1) - 2 \leq Z(H_1).$$

By the minimum degree lower bound we know $\delta - 1 \leq Z(H_2)$, and hence

$$
\begin{aligned}
(2(\delta - 1) - 2) + (\delta - 1) \quad &= \quad 3\delta - 5 \\
&\leq \quad Z(H_1) + Z(H_2) \\
&\leq \quad Z(G) + 1.
\end{aligned}
$$

Rearranging terms, we get our desired result $3\delta - 6 \leq Z(G)$. $\qquad\square$

For graphs with a cut edge and girth at least 5, we can drastically improve the minimum degree lower bound, as illustrated by the following theorem.

Theorem 3.9. *Let G be a graph with minimum degree $\delta \geq 3$, girth $g \geq 5$, and cut edge e. Then,*

$$4\delta - 9 \leq Z(G).$$

Proof. Let G be a graph with minimum degree $\delta \geq 3$, girth $g \geq 5$, and cut edge e such that $G - \{e\} = H_1 \cup H_2$. Since no cut edge of G may lie on a cycle, we know that each component H_1 and H_2 has girth at least 5. Recall $Z(G - e) \leq Z(G) + 1$, from Theorem 3.7. Since zero forcing is additive with respect to disjoint components we have

$$Z(H_1) + Z(H_2) \leq Z(G) + 1.$$

Note that $\delta(H_1) \geq \delta - 1$ and $\delta(H_2) \geq \delta - 1$, since deleting e from G at most reduced the degree of the components by 1. By Theorem 3.6, we know that,

$$2(\delta - 1) - 2 \leq Z(H_i), \ \ i = 1, 2.$$

Hence,

$$
\begin{aligned}
(2(\delta - 1) - 2) + (2(\delta - 1) - 2) \quad &= \quad 4\delta - 8 \\
&\leq \quad Z(H_1) + Z(H_2) \\
&\leq \quad Z(G) + 1.
\end{aligned}
$$

Rearranging, we get our desired result, $4\delta - 9 \leq Z(G)$. $\qquad\square$

The following theorem generalizes the previous two theorems to include k-connected graphs:

Theorem 3.10. *Let G be a graph and suppose there exists a set of vertices $K = \{v_1, ..., v_k\}$ such that $\delta(G \setminus K) \geq 2$ and $G \setminus K$ has induced girth at least 5. Then,*

$$2\delta - 3k - 2 \leq Z(G).$$

Proof. Let K be a set of k vertices whose removal from G forms a subgraph with girth at least 5 and minimum degree at least 2. Next observe that $Z(G \setminus K) \leq Z(G) + k$. Since removing a vertex will most decrease the minimum degree of G by one, removing k vertices will at most reduce the minimum degree of G by k. By Theorem 3.6, we have $2\delta(G \setminus K) - 2 \leq Z(G \setminus K) \leq Z(G) + k$. Rearranging terms we get $2\delta - 3k - 2 \leq Z(G)$. □

4 A General Conjecture

In this section, we make a conjecture regarding improvements to Theorems 3.2 and 3.6. We provide intuitive, numerical, and theoretical justification for these conjectures. We conjecture a lower bound for $Z(G)$ that is dependent on both the girth g and minimum degree δ:

Conjecture 1. *Let G be a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$. Then,*

$$(g-3)(\delta - 2) + \delta \leq Z(G).$$

The intuitive reasoning behind this conjecture is the following. Let G be a graph with girth g and minimum degree δ. Consider any path $v_0 \to v_1 \to \ldots \to v_\ell$ such that v_i forces v_{i+1} for all i and v_0 forces v_1 at $t = 0$. It must be the case that, for all $0 < j < g$ and $0 \leq k \leq g$, v_j is not adjacent to v_k. Hence, v_j requires at least $\delta - 2$ neighbors outside of the path to be colored. Under the assumption that each of these vertices requires a distinct initially colored vertex in order to become colored, each $v_1, v_2, \ldots, v_{g-3}$ require a total of $(\delta - 2)(g - 3)$ initially colored vertices. Note that beginning with $j = g - 2$, the neighbors of v_j may, in fact be the neighbors of v_0 without violating the girth condition. Hence, together with the initial δ vertices needed for v_0 to force v_1 requires at least $\delta + (\delta - 2)(g - 3)$ vertices to be initially colored.

In addition, we can prove the following:

Proposition 4.1. *Given girth $g > 6$, Conjecture 1 is true for sufficiently large minimum degree δ, regardless of n.*

To prove this proposition, we use the *treewidth* parameter, denoted $tw(G)$, as an intermediary. Treewidth is a graph parameter regarding the decomposition of the graph into sets, however, the details are not important for our application.

We apply two results together to prove Proposition 4.1: an exponential lower bound on treewidth in terms of δ and g, and a lower bound on $Z(G)$ in terms of treewidth:

Theorem 4.2 (Chandrana and Subramanian [8])**.** *Let G be a graph with average degree $\bar{d}$, girth g, and treewidth $tw(G)$. Then,*

$$tw(G) \geq \frac{(\bar{d} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)}$$

Theorem 4.3 (Barioli-Barrett-Fallat-Hall-Hogben-Shader-van der Holst [3])**.**

$$tw(G) \leq Z(G)$$

Proof of Proposition 4.1. Given g, choose δ_{min} large enough such that

$$\frac{(\delta_{min} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)} \geq \delta_{min} + (\delta_{min} - 2)(g - 3).$$

This is guaranteed for $g > 6$ as the left side has polynomial degree at least 2 in δ. In which case, by the previous two theorems, we have

$$
\begin{aligned}
\delta_{min} + (\delta_{min} - 2)(g - 3) &\leq \frac{(\delta_{min} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g+1)} \\
&\leq \frac{(\bar{d} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g+1)} \\
&\leq tw(G) \\
&\leq Z(G).
\end{aligned}
$$

Hence, given g, the conjecture holds true for all graphs with $\delta > \delta_{min}$. □

We are able to numerically verify our conjecture for a large number of graphs. However, considering that calculating the zero forcing number of a graph is NP-hard [9], it is difficult to test. We write a compact algorithm in *Mathematica* in order to compute $Z(G)$ using a brute-force approach. In turn, we are able to confirm Conjecture 1 for all 542 triangle-free graphs in the *Wolfram* database with between 9 and 22 vertices. In fact, our computation shows that for more than 37% of these graphs, the conjecture is sharp. Conjecture 1 appears to be sharp for several interesting graphs including the Petersen Graph, the Heawood Graph and several families of graphs as well including cycles, $2 \times 2 \times k$ grids, complete bipartite graphs, among others.

Furthermore, taking $g = 4$ in Conjecture 1, yields a conjecture this is similar in nature to the results presented in Section 3:

Conjecture 2. *Let G be a triangle-free graph with minimum degree $\delta \geq 2$. Then,*

$$
2\delta - 2 \leq Z(G).
$$

5   Concluding Remarks

The results presented in this paper show a link between the girth and minimum zero forcing sets. In particular, the theorems presented provide evidence in favor to Conjectures 1 and 2. Further, the connection between minimum degree and minimum zero forcing sets remain, since the minimum degree appears in all of presented theorems. Future work will include working toward resolving Conjectures 1 and 2, but also will include finding new lower bounds on the zero forcing number in terms of simple graph parameters such as degree, number of edges, and number of vertices.

References

[1] AIM Special Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, Volume 428 (7): 1628–1648, 2008.

[2] D. Amos, Y. Caro, R. Davila, and R. Pepper. Upper bounds on the k-forcing number of a graph. *Discrete Applied Mathematics*, Volume 181: 1–10, 2015.

[3] F. Barioli, W. Barrett, S.M. Fallat, T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst, Parameters Related to Tree-Width, Zero Forcing, and Maximum Nullity of a Graph. *Journal of Graph Theory*, Volume 72 (2): 146–177, 2013.

[4] F. Barioli, W. Barrett, S.M. Fallat, T. Hall, L. Hogben, and H. van der Holst. On the graph complement conjecture for minimum rank. *Linear Algebra and its Applications*, Volume 436 (12): 4373–439, 2012.

[5] R. Brualdi, L. Hogben, and B. Shader. Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns Final Report: Mathematical Results (revised). *American Institute of Mathematics*, 2007.

[6] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Physical review letters*, Volume 99 (10): 100501, 2007.

[7] D. Burgarth, V. Giovannetti, L. Hogben, S. Severini, and M. Young. Logic circuits from zero forcing. arXiv preprint arXiv:1106.4403, 2011.

[8] S. Chandran and C. Subramanian. Girth and treewidth. *Journal of combinatorial theory, Series B*, Volume 93 (1): 23–32, 2005.

[9] C. Chekuri and N. Korula. A graph reduction step preserving element-connectivity and applications. *Automata, Languages and Programming*, 254–265. Springer 2009.

[10] L. Deaett. The minimum semidefinite rank of a triangle-free graph. *Linear Algebra and its Applications*, Volume 434 (8): 1945–1955, 2011.

[11] C. Edholm, L. Hogben, J. LaGrange, and D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. *Linear Algebra and its Applications*, Volume 436 (12): 4352–4372, 2012.

[12] L. Eroh, C. Kang, and E. Yi. Metric Dimension and Zero Forcing Number of Two Families of Line Graphs. *arXiv preprint arXiv:1207.6127*, 2012.

[13] T. Haynes, S. Hedetniemi, S. Hedetniemi, and M. Henning. Domination in graphs applied to electric power networks. *SIAM Journal on Discrete Mathematics*, Volume 15 (4): 519–529, 2002.

[14] S. Meyer. Zero forcing sets and bipartite circulants. *Linear Algebra and its Applications*, Volume 436 (4): 888–900, 2012.

[15] D. Row. Zero forcing number: Results for computation and comparison with other graph parameters (PhD Thesis) *Iowa State University*, 2011.

[16] M. Trefois and J.C. Delvenne. Zero forcing sets, constrained matchings and minimum rank. *Linear and Multilinear Algebra*, 2013.

[17] M. Zhao, L. Kang, and G. Chang. Power domination in graphs. *Discrete Mathematics*, Volume 306 (15): 1812–1816, 2006.