A Generalization of the Euler-Glaisher Partition Bijection

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A Generalization of the Euler-Glaisher Bijection

Andrew Sills

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A partition $\lambda$ of the integer $n$ is a representation of $n$ as an unordered sum of positive integers

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$
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Each summand $\lambda_i$ is called a part of the partition $\lambda$. 
A partition $\lambda$ of the integer $n$ is a representation of $n$ as an unordered sum of positive integers

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$ 

Each summand $\lambda_i$ is called a part of the partition $\lambda$.

Often a canonical ordering of parts is imposed:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r.$$
The Partitions of 6

6  5 + 1  4 + 2  4 + 1 + 1  3 + 3  3 + 2 + 1
3 + 1 + 1 + 1  2 + 2 + 2  2 + 2 + 1 + 1  2 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1 + 1
Euler’s partition identity
Euler’s partition identity

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.
Euler’s partition identity—Example

Of the eleven partitions of 6, four of them have only odd parts:

\[ 5 + 1 \quad 3 + 3 \quad 3 + 1 + 1 + 1 \quad 1 + 1 + 1 + 1 + 1 + 1 + 1 \]
Euler’s partition identity—Example

Of the eleven partitions of 6, four of them have only odd parts:

\[5 + 1 \quad 3 + 3 \quad 3 + 1 + 1 + 1 \quad 1 + 1 + 1 + 1 + 1 + 1 + 1\]

and four of them have distinct parts:

\[6 \quad 5 + 1 \quad 4 + 2 \quad 3 + 2 + 1.\]
Any partition $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r$ may be written in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \ldots,$$
Any partition $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r$ may be written in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \ldots,$$

or more briefly, as

$$\{f_1, f_2, f_3, f_4, \ldots\},$$

where $f_i$ represents the number of appearances of the positive integer $i$ in the partition.
For example, the partition

\[ 6 + 6 + 6 + 6 + 4 + 4 + 3 + 2 + 2 + 2 + 2 + 1 + 1 \]
Frequency Notation for Partitions

For example, the partition

\[ 6 + 6 + 6 + 6 + 4 + 4 + 3 + 2 + 2 + 2 + 2 + 1 + 1 \]

\[ = 2 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + 0 \cdot 5 + 4 \cdot 6 + 0 \cdot 7 + 0 \cdot 8 + 0 \cdot 9 + \ldots \]
For example, the partition

\[ 6 + 6 + 6 + 6 + 4 + 4 + 3 + 2 + 2 + 2 + 2 + 1 + 1 \]

\[ = 2 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + 0 \cdot 5 + 4 \cdot 6 + 0 \cdot 7 + 0 \cdot 8 + 0 \cdot 9 + \ldots \]

may be represented by the frequency sequence

\[ \{2, 4, 1, 2, 0, 4, 0, 0, 0, 0, 0, 0, \ldots \} \].
Thus each sequence \( \{f_i\}_{i=1}^{\infty} \), where each \( f_i \) is a nonnegative integer and only finitely many of the \( f_i \) are nonzero, represents a partition of the integer \( \sum_{i=1}^{\infty} i f_i \).
Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ odd parts.
Glaisher’s proof of Euler’s identity

- Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ odd parts.
- Rewrite $\lambda$ in the form

$$f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots.$$
Glaisher’s proof of Euler’s identity

- Let \( \lambda_1 + \lambda_2 + \cdots + \lambda_r \) be a partition \( \lambda \) of some positive integer \( n \) into \( r \) odd parts.
- Rewrite \( \lambda \) in the form
  \[
  f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots.
  \]
- Replace each \( f_i \) with its binary expansion
  \[
  \cdots + a_{i3} \cdot 8 + a_{i2} \cdot 4 + a_{i1} \cdot 2 + a_{i0} \cdot 1.
  \]
Glaisher’s proof of Euler’s identity

So,

\[ f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots \]
So,

\[ f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots \]

\[ = (\cdots + a_{1,3} \cdot 8 + a_{1,2} \cdot 4 + a_{1,1} \cdot 2 + a_{1,0} \cdot 1) \cdot 1 \]

\[ + (\cdots + a_{3,3} \cdot 8 + a_{3,2} \cdot 4 + a_{3,1} \cdot 2 + a_{3,0} \cdot 1) \cdot 3 \]

\[ + (\cdots + a_{5,3} \cdot 8 + a_{5,2} \cdot 4 + a_{5,1} \cdot 2 + a_{5,0} \cdot 1) \cdot 5 \]

\[ : \]
Glaisher’s proof of Euler’s identity

So,

\[ f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots \]

\[ = (\cdots + a_{1,3} \cdot 8 + a_{1,2} \cdot 4 + a_{1,1} \cdot 2 + a_{1,0} \cdot 1) \cdot 1 \]

\[ + (\cdots + a_{3,3} \cdot 8 + a_{3,2} \cdot 4 + a_{3,1} \cdot 2 + a_{3,0} \cdot 1) \cdot 3 \]

\[ + (\cdots + a_{5,3} \cdot 8 + a_{5,2} \cdot 4 + a_{5,1} \cdot 2 + a_{5,0} \cdot 1) \cdot 5 \]

\[ \vdots \]

\[ = a_{1,0} + 2a_{1,1} + 3a_{3,0} + 4a_{1,2} + 5a_{5,0} + 6a_{3,1} + 7a_{7,0} + \cdots \]

where each \(a_{i,j} \in \{0, 1\}\).
Euler’s partition identity

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.
Euler’s partition identity

The number of partitions of $n$ into nonmultiples of 2 equals the number of partitions of $n$ where no part appears more than once.
Glaisher’s partition identity

The number of partitions of $n$ into nonmultiples of $m$ equals the number of partitions of $n$ where no part appears more than $m - 1$ times.
Proof of Glaisher’s identity
Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ nonmultiples of $m$. 
Proof of Glaisher’s identity

Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ nonmultiples of $m$.

Rewrite $\lambda$ in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots,$$

where $f_i = 0$ if $m \mid i$. 
Proof of Glaisher’s identity

- Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ nonmultiples of $m$.
- Rewrite $\lambda$ in the form
  \[ f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots, \]
  where $f_i = 0$ if $m \mid i$.
- Replace each $f_i$ with its base $m$ expansion
  \[ \cdots + a_{i3} \cdot m^3 + a_{i2} \cdot m^2 + a_{i1} \cdot m + a_{i0} \cdot 1. \]
So,

\[ f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \cdots \]
Proof of Glaisher’s identity

So,

\[
f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \cdots
\]

\[
= \left( \cdots + a_{1,3} \cdot m^3 + a_{1,2} \cdot m^2 + a_{1,1} \cdot m + a_{1,0} \cdot 1 \right) \cdot 1
\]

\[
+ \left( \cdots + a_{2,3} \cdot m^3 + a_{2,2} \cdot m^2 + a_{2,1} \cdot m + a_{2,0} \cdot 1 \right) \cdot 2
\]

\[
+ \left( \cdots + a_{3,3} \cdot m^3 + a_{3,2} \cdot m^2 + a_{3,1} \cdot m + a_{3,0} \cdot 1 \right) \cdot 3
\]

\[
\vdots
\]

where each \( 0 \leq a_{i,j} \leq m - 1. \)
Informal Definition A partition ideal $C$ is a set of partitions such that for each $\lambda \in C$, if one or more parts is removed from $\lambda$, the resulting partition is also in $C$. 
Let \( R_1(n) \) denote the number of partitions of \( n \) into parts congruent to \( \pm 1 \pmod{5} \).
Let $R_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$.

Let $R_2(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_r$ of $n$ such that $\lambda_i - \lambda_{i+1} \geq 2$. 
Let $R_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$.

Let $R_2(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_r$ of $n$ such that $\lambda_i - \lambda_{i+1} \geq 2$.

Then $R_1(n) = R_2(n)$ for all $n$. 
The partitions enumerated by $R_1(n)$ are those for which $f_i = 0$ whenever $i \not\equiv \pm 1 \pmod{5}$.
The First Rogers-Ramanujan Identity

- The partitions enumerated by $R_1(n)$ are those for which $f_i = 0$ whenever $i \not\equiv \pm 1 \pmod{5}$.
- The partitions enumerated by $R_2(n)$ are those for which $f_i + f_{i+1} \leq 1$. 
Let the sequence \( \{d_1^C, d_2^C, d_3^C, \ldots \} \) be defined by

\[
d_j^C = \sup_{\{f_i\}_{i=1}^{\infty} \in C} f_j,
\]
Let the sequence \( \{d_1^C, d_2^C, d_3^C, \ldots \} \) be defined by

\[
d_j^C = \sup_{\{f_i\}_{i=1}^\infty \in C} f_j,
\]

Each \( d_i \) is a nonnegative integer or \(+\infty\).
Let $\mathcal{O}$ denote the set of all partitions with only odd parts.
Let $\mathcal{O}$ denote the set of all partitions with only odd parts.

\[ \{d'_j^\mathcal{O}\}_{j=1}^\infty = \{\infty, 0, \infty, 0, \infty, 0, \ldots\} \]
Let $\mathcal{O}$ denote the set of all partitions with only odd parts.

\[
\{d^\mathcal{O}_j\}_{j=1}^\infty = \{\infty, 0, \infty, 0, \infty, 0, \ldots \}
\]

Let $\mathcal{D}$ denote the set of all partitions with distinct parts.
Let \( \mathcal{O} \) denote the set of all partitions with only odd parts.

\[
\{ d_{j_1}^{\mathcal{O}} \}_{j_1=1}^\infty = \{ \infty, 0, \infty, 0, \infty, 0, \ldots \}
\]

Let \( \mathcal{D} \) denote the set of all partitions with distinct parts.

\[
\{ d_{j_1}^{\mathcal{D}} \}_{j_1=1}^\infty = \{ 1, 1, 1, 1, 1, 1, \ldots \}.
\]
Let $p(C, n)$ denote the number of partitions of an integer $n$ in the partition ideal $C$. 
Equivalent Partition Ideals

- Let $p(C, n)$ denote the number of partitions of an integer $n$ in the partition ideal $C$.

- We say that two partition ideals $C_1$ and $C_2$ are equivalent, and write $C_1 \sim C_2$, if $p(C_1, n) = p(C_2, n)$ for all integers $n$. 
Euler’s Partition Identity

\[ O \sim D. \]
Define the multiset associated with $C$, $M(C)$, as follows:

$$M(C) := \{ j(d^C_j + 1) \mid j \in \mathbb{Z}_+ \text{ and } d^C_j < \infty \}.$$
Define the multiset associated with $C$, $M(C)$, as follows:

$$M(C) := \{ j(d_j^C + 1) \mid j \in \mathbb{Z}_+ \text{ and } d_j^C < \infty \}.$$ 

Andrews proved $C_1 \sim C_2$ if and only if $M(C_1) = M(C_2)$. 
\[ M(\mathcal{O}) = M(\mathcal{D}) = \{2, 4, 6, 8, 10, 12, \ldots \} \]
MacMahon defined a *partition of infinity* to be a formal expression of the form

\[(g_1 - 1) \cdot 1 + (g_2 - 1) \cdot g_1 + (g_3 - 1) \cdot (g_1 g_2) + (g_4 - 1) \cdot (g_1 g_2 g_3) + \ldots\]

where

- each \(g_i \geq 2\)
- or for some fixed \(k\),
  - \(g_1, g_2, g_3, \ldots, g_{k-1} > 1\),
  - \(g_k = \infty\), and
  - \(g_{k+1} = g_{k+2} = g_{k+3} = \ldots = 1\).
Note that a partition of infinity may be thought of as a minimal bounding sequence for a partition ideal of order one with

\[ d_1 = g_1 - 1 \]
\[ d_{g_1} = g_2 - 1 \]
\[ d_{g_1g_2} = g_3 - 1 \]
\[ d_{g_1g_2g_3} = g_4 - 1 \]
\[ \vdots \]

and

\[ d_i = 0 \text{ if } i \not\in \{1, g_1, g_1g_2, g_1g_2g_3, \ldots \} \].
If $g_i = m$ for all $i$, we have a minimal bounding sequence for the partition ideal of the “base $m$ expansion.”
All partitions of infinity have generating function

\[ \frac{1}{1 - q}. \]
Given any partition ideal of order 1 $C$ for which each term in the minimal bounding sequence is 0 or $\infty$, 

Given any partition ideal of order 1 $C$ for which each term in the minimal bounding sequence is 0 or $\infty$, and any equivalent partition ideal of order 1 $C'$,
Given any partition ideal of order 1 $C$ for which each term in the minimal bounding sequence is 0 or $\infty$,

and any equivalent partition ideal of order 1 $C'$,

there exists a collection of partitions of infinity which gives rise to a “Glaisher-type bijection” from $C$ to $C'$.
Given any partition ideal of order 1 \( C \) for which each term in the minimal bounding sequence is 0 or \( \infty \),

and any equivalent partition ideal of order 1 \( C' \),

there exists a collection of partitions of infinity which gives rise to a “Glaisher-type bijection” from \( C \) to \( C' \).

Further, there is an explicit algorithm for finding the required partitions of infinity.
Let $C$ denote the set of partitions into parts not equal to 2, 9, 10, 12, 18 or 20.
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Let $C'$ denote the set of partitions where the parts

- 1, 9, and 10 may appear at most once,
- 3 and 4 may appear at most twice,
- 2 may appear at most four times,
- and all other positive integers may appear without restriction.
An Inelegant Partition Identity

Let $C$ denote the set of partitions into parts not equal to 2, 9, 10, 12, 18 or 20.

Let $C'$ denote the set of partitions where the parts
- 1, 9, and 10 may appear at most once,
- 3 and 4 may appear at most twice,
- 2 may appear at most four times,
- and all other positive integers may appear without restriction.

$C \sim C'$. 
\[ d_j^C = \begin{cases} 
0 & \text{if } j \in \{2, 9, 10, 12, 18, 20\} \\
\infty & \text{otherwise.} 
\end{cases} \]

\[ \left\{ d_j^{C'} \right\}_{j=1}^{\infty} = \{1, 4, 2, 2, \infty, \infty, \infty, \infty, 1, 1, \infty, \infty, \infty, \infty, \ldots \}. \]
An Inelegant Partition Identity

$\mathbb{d}_{j}^{C} = \begin{cases} 
0 & \text{if } j \in \{2, 9, 10, 12, 18, 20\} \\
\infty & \text{otherwise.}
\end{cases}$

$\{\mathbb{d}_{j}^{C'}\}_{j=1}^{\infty} = \{1, 4, 2, 2, \infty, \infty, \infty, 1, 1, \infty, \infty, \infty, \infty, \ldots \}.$

Any partition of $n$ in $C$ can be written in the form

$$n = f_{1} \cdot 1 + \sum_{i=3}^{8} f_{i} \cdot i + f_{11} \cdot 11 + \sum_{i=13}^{17} f_{i} \cdot i + f_{19} \cdot 19 + \sum_{i=21}^{\infty} f_{i} \cdot i,$$

where each $f_{i}$ is a nonnegative integer.
Expand $f_1$ by the partition of infinity defined by $g_{1,1} = 2$, $g_{1,2} = 5$, $g_{1,3} = 2$, $g_{1,4} = \infty$, $g_{1,k} = 1$ if $k > 4$. 
Expand $f_1$ by the partition of infinity defined by $g_{1,1} = 2$, $g_{1,2} = 5$, $g_{1,3} = 2$, $g_{1,4} = \infty$, $g_{1,k} = 1$ if $k > 4$.

Expand $f_3$ by the partition of infinity defined by $g_{3,1} = 3$, $g_{3,2} = 2$, $g_{3,3} = \infty$, $g_{3,k} = 1$ if $k > 3$. 
Expand $f_1$ by the partition of infinity defined by $g_{1,1} = 2, g_{1,2} = 5, g_{1,3} = 2, g_{1,4} = \infty, g_{1,k} = 1$ if $k > 4$.

Expand $f_3$ by the partition of infinity defined by $g_{3,1} = 3, g_{3,2} = 2, g_{3,3} = \infty, g_{3,k} = 1$ if $k > 3$.

Expand $f_4$ by the partition of infinity defined by $g_{4,1} = 3, g_{4,2} = \infty, g_{4,k} = 1$ if $k > 2$. 
\[ n = \left( a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(2 \cdot 5) + a_{1,3}(2 \cdot 5 \cdot 2) \right) 1 \]
\[ + \left( a_{2,0}(1) + a_{2,1}(3) + a_{2,2}(3 \cdot 2) \right) 3 \]
\[ + \left( a_{4,0}(1) + a_{4,1}(3) \right) 4 \]
\[ + \left( a_{5,0}(1) \right) 5 \]
\[ + \left( a_{6,0}(1) \right) 6 \]
\[ + \left( a_{7,0}(1) \right) 7 \]
\[ + \left( a_{8,0}(1) \right) 8 \]
\[ + \left( a_{11,0}(1) \right) 11 \]
\[ \vdots \]

where \( 0 \leq a_{j,k} \leq g_{j,k+1} - 1 = d_j g_{j,1} g_{j,2} \cdots g_{j,k} \).
Apply the distributive property to obtain

\[ n = a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(10) + a_{1,4}(20) + a_{2,0}(3) + a_{2,1}(9) + a_{2,2}(18) + a_{4,0}(4) + a_{4,1}(12) + a_{5,0}(5) + a_{6,0}(6) \]

where, in particular,

\[ a_{1,0} \leq 1, \quad a_{1,1} \leq 4, \quad a_{1,2} \leq 2, \quad a_{2,0} \leq 2, \quad a_{2,1} \leq 1, \quad a_{4,0} \leq 2. \]
Any $C$ for which each term in $\{d_j^C\}_{j=1}^{\infty}$ is 0 or $\infty$ has an $M(C)$ with no repeated elements.
Any $C$ for which each term in $\{d_j^C\}_{j=1}^\infty$ is 0 or $\infty$ has an $M(C)$ with no repeated elements.

If for some $C'$, $M(C')$ has no repeated elements, it is equivalent to some $C$ with minimal bounding sequence consisting of only 0’s and $\infty$’s.
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If for some $C'$, $M(C')$ has no repeated elements, it is equivalent to some $C$ with minimal bounding sequence consisting of only 0’s and $\infty$’s.

This $C$ maps bijectively to $C'$ via the Glaisher-type bijection where the role of the base $m$ expansion is played by an appropriate partition of infinity.
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This $C$ maps bijectively to $C'$ via the Glaisher-type bijection where the role of the base $m$ expansion is played by an appropriate partition of infinity.

If $C' \sim C''$ and $M(C') = M(C'')$ has no repeated elements, there is a canonical bijection between them.