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Scheduling N Burgers for a k-Burger Grill: Chromatic Numbers With Restrictions

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Abstract

The chromatic number has a well-known interpretation in the area of scheduling. If the vertices of a finite, simple graph are committees, and adjacency of two committees indicates that they must never be in session simultaneously, then the chromatic number of the graph is the smallest number of hours during which the committees/vertices of the graph may all have properly scheduled meetings of one continuous hour each. Slivnik [3] showed that the fractional chromatic number can be similarly characterized. In that characterization, the meetings are allowed to be broken into a finite number of disjoint intervals.

Here we consider chromatic numbers definable similarly, with the additional restriction that there are only k meeting rooms available; that is, at each instant no more than k committees can be in session.

Keywords and phrases: fractional chromatic number, independent set, vertex independence number

1 Cooking many burgers on a small grill

The seminal problem from which this inquiry grew was cooked up over a dozen years ago by the second author: assuming that each hamburger patty must be grilled for one unit of time on each side, what is the minimum number of time units needed to grill 3 burgers on a 2-burger grill (a grill on which only two burgers will fit at one time)? It's not hard to find a way of completing the task in 3 time units, during which each patty side gets exactly one time unit of grilling, and the grill is in full use (two burgers on) at all times—assuming flipping and moving patties require no time at all. It is intuitively clear that 3 time units is the minimum called for, but a satisfactory proof requires more than intuition.

The real question behind this problem is not about grilling burgers, important as that is, but rather: what is an interesting general problem of which these burger-grilling problems are special cases? It is the question that we can now answer.

All graphs here are finite and simple. A *scheduling function* on a graph G is a function φ on $V(G)$ such that for all u, $v \in V(G)$,

(i) $\varphi(v)$ is a finite union of open intervals on the real line;

(ii) if $uv \in E(G)$ then $\varphi(u) \cap \varphi(v) = \emptyset$; and

(iii) the *measure* of $\varphi(v)$, i.e., the sum of lengths of its maximal subintervals, is 1.

The idea here has to do with an issue briefly discussed in the Abstract. If the vertices of G are committees such that $uv \in E(G)$ implies that committees u and v must never be in session simultaneously, then a scheduling function gives a proper coloring of G with meeting times for these committees, such that each committee gets a total of one unit of meeting time. Some may find it disturbing that these meetings can be broken into intervals separated from each other. Clearly it is impractical to allot a committee 45 minutes on one fine morning and then 13 seconds some time later. We will have some consolation on this issue later in this paper, but we are leaving most of that territory unexplored.

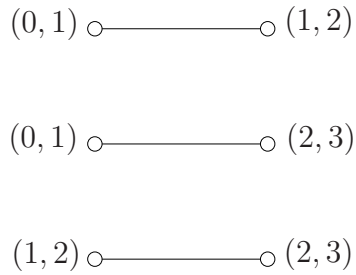

Figure 1: Scheduling 3 burgers for a 2-burger grill

Suppose that k is a positive integer. A scheduling function φ on G *obeys restriction* k if and only if, for each real number t, $|\{v \in V(G) \mid t \in \varphi(v)\}| \leq k$. Note that because of requirement (ii) in the definition of scheduling function, if φ is such a function on G and $t \in \mathbb{R}$, then $\{v \in V(G) \mid t \in \varphi(v)\}$ is an independent set of vertices in G. Therefore, if $k \geq \alpha(G) =$ vertex independence number of G, then restriction k is no restriction at all; it is obeyed by every scheduling function on G.

In actual scheduling of committee meetings, obedience to restriction k means that no more than k meetings may every be in session simultaneously. Clearly such a constraint may arise naturally; there may be only k meeting rooms available. In the problem of 3 burgers on a 2-burger grill, the graph involved is $3K_2$, the disjoint union of 3 edges. The 6 vertices are the 6 sides of the 3 hamburgers, with two vertices adjacent if and only if they are the two sides of the same burger patty. The restriction that our scheduling functions on this graph must obey is $k = 2$, arising from the size of the grill.

The fractional chromatic number, χ_f, has many equivalent definitions, most of which can be found in [2]; but here is one, due to T. Slivnik [3], which does not appear there:
$\chi_f(G) = \inf[T > 0;$ there is a scheduling function φ on G such that for each $v \in V(G), \varphi(v) \subseteq (0, T)]$. (It is not immediately obvious, but the inf is min for every G.)

For a graph G and a positive integer k, we define the fractional chromatic number of G with restriction k to be $\chi_f(G; k) = \inf[T > 0;$ there is a scheduling function φ on G which obeys restriction k such that for each $v \in V(G), \varphi(v) \subseteq (0, T)]$ (We shall see later that the inf in this definition is a min, which, in view of the obvious fact that $\chi_f(G; k) = \chi_f(G)$ for all $k \geq \alpha(G)$, will establish the same for Slivnik's characterization of $\chi_f(G)$.)

Two other obvious observations regarding the restricted fractional chromatic number: $\chi_f(G; 1) = |V(G)|$ for all graphs G, and

$$\chi_f(G; k) \geq \chi_f(G; k + 1)$$

for all graphs G and positive integers k.

The 3-burgers-on-a-two-burger-grill problem is to determine $\chi_f(3K_2; 2)$. As the reader has probably already worked out, we can see that $\chi_f(3K_2; 2) \leq 3$ by the scheduling function indicated in Figure 1, which obeys restriction 2 and finishes the burgers in 3 time units:

But how to give a clean proof of the obvious, that $\chi_f(3K_2; 2) \geq 3$?

For a graph G and a positive integer k, we define $\alpha(G; k) = \min(k, \alpha(G))$.

Proposition 1.1. *Suppose that G is a graph and k is a positive integer. Then $\chi_f(G; k) \geq \frac{|V(G)|}{\alpha(G;k)}$.*

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Proof. Suppose that φ is a scheduling function on G, obeying restriction k, with $\varphi(v) \subseteq (0,T)$ for all $v \in V(G)$. For each $u \in V(G)$, let f_u denote the characteristic function of $\varphi(u)$; that is, $f_u(t) = \{ \begin{array}{ll} 1 & \text{if } t \in \varphi(u) \\ 0 & \text{otherwise.} \end{array}$ Because φ is a scheduling function on G obeying restriction k, with $\varphi(u) \subseteq (0,T)$ for all $u \in V(G)$, we have

$$\begin{aligned} |V(G)| &= \textstyle\sum_{u \in V(G)} (\text{measure of } \varphi(u)) \\ &= \textstyle\sum_{u \in V(G)} \int_0^T f_u(t)dt \\ &= \int_0^T (\textstyle\sum_{u \in V(G)} f_u(t))dt \\ &\leq \int_0^T \alpha(G;k)dt = \alpha(G;k)T. \end{aligned}$$

□

Therefore $\chi_f(3K_2;2) \geq \frac{6}{2} = 3$, confirming intuition. We will soon see that $\chi_f(NK_2;k) = \frac{2N}{k}$ for all $k \leq N = \alpha(NK_2)$, and all positive integers N. The proof will show how to schedule N burgers for a k-burger grill, $k \leq N$, so that all the cooking is done in $\frac{2N}{k}$ time units.

First, an improvement of the bound on $\chi_f(G;k)$ in Proposition 1.1. The *Hall ratio* of a graph G is

$$\rho(G) = \max\left[\frac{|V(H)|}{\alpha(H)}; H \text{ is an induced subgraph of } G\right].$$

This a well known lower bound on $\chi_f(G)$. We define the *Hall ratio with restriction* k to be

$$\rho(G;k) = \max\left[\frac{|V(H)|}{\alpha(H;k)}; H \text{ is an induced subgraph of } G\right]$$

Corollary 1.1. *For any graph G and positive integer k, $\chi_f(G;k) \geq \rho(G;k)$.*

Proof. For any subgraph H of G, the restriction to $V(H)$ of any scheduling function on G which obeys restriction k is a scheduling function on H which obeys restriction k. Therefore

$$\chi_f(G;k) \geq \chi_f(H;k) \geq \frac{|V(H)|}{\alpha(H;k)}.$$

Taking the max over all (induced) H on the right gives the result. □

We shall finish this section by showing that $\chi_f(NK_p;k) = \frac{Np}{k}$ for all positive integers N, p, and $k \leq N = \alpha(NK_p)$. Further, our proof will show that for all such N, p, k, a scheduling function φ obeying restriction k can be found for NK_p such that each set $\varphi(v)$ is the union of no more than 2 open intervals; we can make $\varphi(v)$ a single open interval for each vertex v if and only if k divides Np.

Lemma 1.2. *Suppose that a, b, c, d are positive integers such that $c \leq b$ and $ac = bd$. Then there is a simple bipartite graph with bipartition $A, B, |A| = a$, $|B| = b$, such that each vertex in A has degree c, and each vertex in B has degree d.*

Proof. Note that $c \leq b$ and $ac = bd$ imply that $a \geq d$. Let the vertices of A be $u_1, \ldots, u_a$, and the vertices of B be $1, \ldots, b$, thought of as the congruence classes in the integers mod b. Let u_i be adjacent to the congruence classes $(i-1)c+1, \ldots, ic, \mod b$. Since $c \leq b$, these are distinct, so there are no double edges in the bipartite graph thus defined. Each vertex in A has degree c in this graph. As r varies from 0 to $a-1$ and t varies from 1 to c, independently, the integer $rc+t$ varies injectively over $1, \ldots, ac = bd$. Therefore each congruence class mod b is wandered over exactly d times; that is, each vertex in B has degree d. □

Corollary 1.3. *(of the proof of Lemma 1.2) Let a, b, c, d be as in Lemma 1.2, and let H be the bipartite graph described in the proof of Lemma 1.2, with bipartition A, B. Let the vertices of $B = \{1, \ldots, b\}$ be thought of as ordinary integers. Then for each $u \in A$, $N_H(u)$ is either a block $\{t, \ldots, t+c-1\}$, for some t satisfying $1 \leq t \leq b-c+1$, of consecutive integers, or the union $\{t, \ldots, b\} \cup \{1, \ldots, c+t-(b+1)\}$, for some t satisfying $b-c+2 \leq t \leq b$, of two blocks of consecutive integers.*

Theorem 1.4. *Suppose that $k \leq N$ and p are positive integers, and $G = NK_p$. Then $\chi_f(G;k) = \frac{Np}{k}$. Further, an optimal scheduling function φ on G which obeys restriction k can be found such that for every $v \in V(G)$, $\varphi(v)$ is the union of no more than two intervals; such a φ can be found such that $\varphi(v)$ is a single interval (of length 1) for each $v \in V(G)$ if and only if k divides Np.*

Proof. We construct a bipartite graph H as in the proof of Lemma 1.2 with bipartition A, B, satisfying $|A| = N$, $|B| = Np$, with every vertex of A having degree kp and every vertex of B having degree k. It is easy to see that the hypotheses of Lemma 1.2 are satisfied.

By appeal to Corollary 1.3 and the proof of Lemma 1.2, each neighbor set of a vertex $v \in A$ is a list of kp consecutive congruence classes mod $(b = Np)$. Divide this list into p sublists of k consecutive congruence classes mod b. All but, possibly, one of these will be a block of consecutive integers in the labeling of B as $\{1, \ldots, b\}$ and any one sublist not so favored will be the union of two sublists, $\{t, \ldots, b\} \cup \{1, \ldots, k+t-(b+1)\}$, for some t satisfying $b-k+2 \leq t \leq b$, of consecutive integers.

Let the N vertices of A correspond to the N cliques K_p of which G is the disjoint union. Let the Np vertices of B correspond to the intervals $(0, \frac{1}{k}), (\frac{1}{k}, \frac{2}{k}), \ldots, (\frac{Np-1}{k}, \frac{Np}{k})$. Let the p sublists of each neighbor set of vertices in A be assigned to the p vertices of the clique to which the vertex corresponds, and let each vertex of G be assigned the union of the intervals to which it is adjacent in H, together with any endpoints that need to be included to make "meeting periods" of each vertex into one open interval or the union of two open intervals. For instance, if a vertex of G is assigned vertices $(\frac{t}{k}, \frac{t+1}{k}), \ldots, (\frac{t+k-1}{k}, \frac{t+k}{k})$ of B, for some $t \in \{0, \ldots, Np-k\}$, then the meeting time assigned to the vertex will be $(\frac{t}{k}, \frac{t}{k}+1)$.

Since degrees in B are all equal to k, restriction k is obeyed by the scheduling function thus defined, and clearly every vertex of G is assigned a meeting time of measure 1.

Since the scheduling function schedules all meetings within $(0, \frac{Np}{k})$, we have $\chi_f(G;k) \leq \frac{Np}{k}$. On the other hand, $\chi_f(G;k) \geq \frac{|V(G)|}{\alpha(G;k)} = \frac{Np}{k}$; therefore $\chi_f(G;k) = \frac{Np}{k}$. [An easy corollary, in passing: $\rho(G;k) = \frac{Np}{k}$, if $k \leq N = \alpha(G)$.]

It is clear that if k divides Np, then the scheduling function described assigns a single interval of length 1 to each vertex of G. Now suppose that there is a scheduling function φ on G, obeying restriction k, which assigns to each $v \in V(G)$ a single subinterval of length 1

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of $(0, \frac{Np}{k})$. Since $k = \alpha(G;k)$ (because $k \leq N = \alpha(G)$) and $|V(G)| = Np$, we see from the proof of Proposition 1.1, with $T = Np/k$, that outside of a set of measure 0 (in fact, outside of a finite set), each point of $(0, \frac{Np}{k})$ must lie in exactly k of these intervals. Therefore, exactly k of these intervals must have left hand endpoint 0, and then exactly k must have left hand endpoint $1/k$, etc. So Np/k is an integer. □

Theorem 1.4 takes care of optimal scheduling of N burgers of any shape, so long as they each have p sides, and each side is to be grilled the same unit of time, with the maximum number of separate grilling intervals (1 or 2) per side minimized, on any size grill. What are not dealt with are

(1) minimizing the number of instances of 2 separated grilling intervals, when $k \nmid Np$, and

(2) when $k \nmid Np$, minimizing the maximum, or perhaps the sum, or perhaps some other measure of an average, of the lengths of intervals between the intervals of a 2-interval grilling. Indeed, the scheduling function described in the proof of Theorem 1.4 gives a worst possible result for (2): if $k \nmid N_p$, the interval between the end of the first session and the begining of the 2nd session, of any 2-session meeting/grilling, is $\frac{Np}{k} - 1$, the maximum possible.

We propose these optimization problems as topics for minds finer than ours.

What if $G = K_{p_1} + \cdots + K_{p_N}$, a disjoint union ("sum") of cliques of orders $p_1, \ldots, p_N$? We will deal with this problem in section 3.

2 Alternative definitions of $\chi_f(G;k)$

In this section we will recapitulate some of the theory of the fractional chromatic number for the restricted fractional chromatic number. For those readers who are familiar with the theory of the fractional chromatic number, the main results of this section will be unsurprising, and their proofs will be recognizable as obvious modifications of standard proofs in the theory of the fractional chromatic number.

But there may be readers with an interest in this paper who are not well-acquainted with the theory of the fractional chromaic number; and even among those who are knowledgeable in this area, the scheduling characterization of the fractional chromatic number may not be terribly familiar. Therefore we think we need to have this hard slog of a section in this paper, but, in conformance with shrewd suggestions of the referee(s) and the editor(s), we are softening the blow by hereby warning readers that they do not necessarily need to pay much attention to this section, and by consigning the more difficult proofs in this section to an Appendix at the paper's end.

If S is the union of a finite number of pairwise disjoint bounded open intervals in $\mathbb{R} = (-\infty, \infty)$, let $m(S)$ denote the number of (maximal) intervals constituting S, and let $\mu(S)$ denote the measure of S, the sum of the lengths of the $m(S)$ intervals whose union is S.

Lemma 2.1. *Suppose that φ is a function on $V(G)$ satisfying all the requirements to be a scheduling function on G except (iii), in place of which φ satisfies*

(iii)' $\mu(\varphi(v)) \geq 1$ *for each* $v \in V(G)$.

Then there is a scheduling function $\tilde{\varphi}$ *on* G *satisfying, for each* $v \in V(G)$, $m(\tilde{\varphi}(v)) = m(\varphi(v))$ *and* $\tilde{\varphi}(v) \subseteq \varphi(v)$.

Proof. For each $v \in V(G)$ such that $\mu(\varphi(v)) = 1$, set $\tilde{\varphi}(v) = \varphi(v)$. If $\mu(\varphi(v)) > 1$, shrink some or all of the intervals whose union is $\varphi(v)$ to non-empty subintervals with union $\tilde{\varphi}(v)$ satisfying $\mu(\tilde{\varphi}(v)) = 1$. □

For a positive integer k and a graph G, let

$$\mathcal{I}_k(G) = \{S \subseteq V(G) \mid S \text{ is independent in } G \text{ and } |S| \leq k\}.$$

If $k \geq \alpha(G)$, we simplify: $\mathcal{I}_k(G) = \mathcal{I}(G)$, the collection of all independent sets of vertices in G. If G is fixed in the discussion we may use $\mathcal{I}_k$ to stand for $\mathcal{I}_k(G)$.

A *fractional proper coloring of* G *satisfying restriction* k is a function $f : \mathcal{I}_k(G) \to [0, \infty)$ satisfying: for each $v \in V(G)$,

$$\sum_{\{S \in \mathcal{I}_k | v \in S\}} f(S) \ \geq \ 1.$$

Note that if f is a fractional proper coloring of G satisfying restriction k, then so is $\tilde{f}$, defined by $\tilde{f}(S) = \min[1, f(S)]$, $S \in \mathcal{I}_k(G)$.

Theorem 2.2. *For any graph* G *and positive integer* k, $\chi_f(G; k) =$ $\min[\sum_{S \in \mathcal{I}_k(G)} f(S); f$ *is a fractional proper coloring of* G *satisfying restriction* $k]$.

The proofs of Theorem 2.2 and the Corollaries to follow are in the Appendix.

Corollary 2.3 (of Theorem 2.2 and its proof)**.** *The* inf *in the definition of* $\chi_f(G; k)$ *is really a min. Furthermore,* $\chi_f(G; k)$ *is a rational number, and a scheduling function* φ *on* G *obeying restriction* k*, with* $\varphi(v) \subseteq (0, \chi_f(G; k))$ *for all* $v \in V(G)$*, can be found such that for each* $v \in V(G)$ $\varphi(v)$ *is a finite union of open intervals with rational endpoints.*

A *fractional clique* (in G) *satisfying restriction* k is a function h: $V(G) \to [0, \infty)$ such that for all $S \in \mathcal{I}_k(G)$, $\sum_{v \in S} h(v) \leq 1$. Note that the range of any such h is contained in $[0, 1]$. It follows, by an argument similar to that for fractional proper colorings satisfying restriction k, that $\{\sum_{v \in V(G)} h(v) \mid h$ is a fractional clique in a G satisfying restriction $k\}$ has a largest element, which we will call the *fractional clique number with restriction* k and denote $\omega_f(G; k)$.

Remark: there is an infelicity in our terminology which we will just have to live with. A *fractional clique* in G is what we would call here a fractional clique satisfying restriction $\alpha(G)$. But then if $1 \leq k < \alpha(G)$ there will be fractional cliques satisfying restriction k, by our definition, which are not, in fact, fractional cliques. The constant function $1/k$ is one such. Be warned.

Corollary 2.4. $\omega_f(G; k) = \chi_f(G; k)$.

For m, r positive integers, with $m > r$, let $[m] = \{1, \ldots, m\}$ and let $\binom{[m]}{r} = \{S \subseteq [m] \mid |S| = r\}$. An (m, r)*-coloring of* G *satisfying restriction* k is a function $\varphi : V(G) \to \binom{[m]}{r}$ such that

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(i) if $uv \in E(G)$ then $\varphi(u) \cap \varphi(v) = \emptyset$, and

(ii) for each $j \in \{1, \ldots, m\}$,
$$|\{v \mid j \in \varphi(v)\}| \leq k.$$

The r-fold chromatic number of G under restriction k is the smallest m such there is an (m, r)-coloring of G which satisfies restriction k. We denote this number by $\chi^{(r)}(G; k)$.

Theorem 2.5.
$$\begin{aligned}\chi_f(G;k) &= \inf_r \tfrac{1}{r}\chi^{(r)}(G;k)\\ &= \min_r \tfrac{1}{r}\chi^{(r)}(G;k)\\ &= \lim_{r\to\infty} \tfrac{1}{r}\chi^{(r)}(G;k)\end{aligned}$$

The proof of Theorem 2.5 is in the Appendix.

3 Calculating $\chi_f(G; k)$

If G and H are graphs, let $G+H$ denote the disjoint union of G and H, and $G \vee H = \overline{\bar{G} + \bar{H}}$ the *join* of G and H, obtained by taking disjoint copies of G and H, together with all $V(G) - V(H)$ edges.

Theorem 3.1. $\chi_f(G \vee H; k) = \chi_f(G; k) + \chi_f(H; k)$.

Proof. Applying Corollary 2.3, we can suppose that φ and ψ are scheduling functions obeying restriction k on G and H, respectively, with $\bigcup_{v\in V(G)} \varphi(v) \subseteq (0, \chi_f(G; k))$ and $\bigcup_{u\in V(H)} \psi(u) \subseteq (\chi_f(G; k), \chi_f(G; k)+\chi_f(H; k))$. Then θ, defined on $V(G\vee H) = V(G)\dot\cup V(H)$ by $\theta(v) = \varphi(v)$, $v \in V(G)$, $\theta(u) = \psi(u)$, $u \in V(H)$, is a scheduling function on $G \vee H$ obeying restriction k because every independent set of vertices in $G \vee H$ is either a subset of $V(G)$ or of $V(H)$; and for every $w \in V(G \vee H)$, $\theta(w) \subseteq (0, \chi_f(G; k) + \chi_f(H; k))$. Therefore, $\chi_f(G \vee H; k) \leq \chi_f(G; k) + \chi_f(H; k)$.

To prove the reverse inequality, we invoke Theorem 2.2, as well as its proof. Let
$$f : \mathcal{I}_k(G \vee H) = \mathcal{I}_k(G) \dot\cup \mathcal{I}_k(H) \to [0, 1]$$
be a fractional proper coloring of G satisfying restriction k, such that
$$\sum_{S\in\mathcal{I}_k(G\vee H)} f(S) = \sum_{s\in\mathcal{I}_k(G)} f(S) + \sum_{U\in\mathcal{I}_k(H)} f(U) = \chi_f(G \vee H; k).$$
For each $v \in V(G)$, any $S \in \mathcal{I}_k(G \vee H)$ containing v is an element of $\mathcal{I}_k(G)$. Therefore, the restriction of f to $\mathcal{I}_k(G)$ is a fractional proper coloring of G satisfying restriction k; the same statement holds with G replaced by H. Therefore
$$\begin{aligned}\chi_f(G \vee H; k) &= \textstyle\sum_{S\in\mathcal{I}_k(G)} f(S) + \sum_{U\in\mathcal{I}_k(G)} f(U)\\ &\geq \chi_f(G; k) + \chi_f(H; k)\end{aligned}$$

□

For any positive integer m, $\bar{K}_m$ denotes the complement of K_m, the empty graph on m vertices. For $r \geq 2$ and positive integers $m_1, \ldots, m_r$, the complete r-partite graph with parts of sizes $n_1, \ldots, n_r$, denoted $K_{n_1}, \ldots, n_r$, is $\bar{K}_{n_1} \vee \cdots \vee \bar{K}_{n_r} = \overline{K_{n_1} + \cdots + K_{n_r}}$. (It is well understood that both $\vee$ and $+$ can be viewed as commutative and associative binary operations on "unlabeled" graphs, meaning isomorphism classes of graphs.)

Corollary 3.2. *For any positive integers $r \geq 2$, $n_1, \ldots, n_r$, and k,*

$$\chi_f(K_{n_1,\ldots,n_r}; k) = \sum_{j=1}^{r} \max(\frac{n_j}{k}, 1).$$

Proof. The result will follow by induction on r if we show that $\chi_f(\bar{K}_m; k) = \max(\frac{m}{k}, 1)$, for all positive integers m and k. Since $\bar{K}_m = mK_1$, the result follows from Theorem 1.4 when $k \leq m$, and when $k > m = \alpha(\bar{K}_m)$, $\chi_f(\bar{K}_m; k) = \chi_f(\bar{K}_m; m) = 1$. □

Corollary 3.3. *To find a scheduling function φ on $G \vee H$ obeying restriction k with $\varphi(v) \subseteq (0, \chi_f(G \vee H; k))$ for all $v \in V(G \vee H) = V(G) \dot{\bigcup} V(H)$, one can do no better than to find scheduling functions φ_1, φ_2 obeying restriction k on G and H separately, with $\varphi_1(v) \subseteq (0, \chi_f(G; k))$ for all $v \in V(G)$ and $\varphi_2(u) \subseteq (\chi_f(G; k), \chi_f(G; k) + \chi_f(H; k))$ for all $u \in V(H)$; and then to define $\varphi = \begin{cases} \varphi_1 & \text{on } V(G) \\ \varphi_2 & \text{on } V(H) \end{cases}$.*

Therefore, to obtain an optimal scheduling function obeying restriction k for $K_{n_1,\ldots,n_r}$, one can "concatenate" scheduling functions obeying restriction k for each $\bar{K}_{n_i}$, and a recipe for obtaining these is given in the proof of Theorem 1.4 (take $p = 1$, $N = n_i$; if $k > n_i$, replace k by n_i). The scheduling function φ on $G = K_{n_1,\ldots,n_r}$ obtained in this way will have the property that for each $v \in V(G)$, $\varphi(v)$ is the union of no more than two open intervals; if $k \mid n_i$, $i = 1, \ldots, r$, then arrangements can be made so that $\varphi(v)$ is a single open interval, for each $v \in V(G)$. We doubt that $k \mid n_i$, $i = 1, \ldots, r$, is a necessary condition for each committee to be assigned a single open interval by an optimal scheduling function, but we shall leave the question open for now.

We leave it as an enjoyable exercise to verify that if $G = K_{n_1,\ldots,n_r}$, then $\rho(G; k) = \max(r, \frac{n_1 + \cdots + n_r}{k})$. We mention this in order to point out that $\rho(G; k) = \chi_f(G; k)$ if and only if either $k \leq \min_i n_i$ or $\alpha(G) = \max_i n_i \leq k$.

Every graph G is a spanning subgraph of some $K_{n_1,\ldots,n_r}$, $r = \chi(G)$, and therefore Corollary 3.2 supplies upper estimates of $\chi_f(G; k)$ for all k. Of course, $\chi_f(G; 1) = |V(G)|$ is known, and therefore uninteresting: it is for $2 \leq k \leq \alpha(G)$ that we are after $\chi_f(G; k)$.

For instance, if P is the Petersen graph, by properly coloring P with 3 colors we see that P is a subgraph of $K_{3,3,4}$ and , therefore, that $\chi_f(P, k) \leq 2\max(1, \frac{3}{k}) + \max(1, \frac{4}{k})$. For $k = 1$, we have, as is always the case, $\chi_f(P; 1) = |V(P)| = |V(K_{3,3,4})| = 10$. But for $k = 2, 3$, we also have equality by Corollary 2.4: the constant function $1/2$ on $V(P)$ is a fractional clique in P satisfying restriction 2, so

$$\begin{aligned} \frac{|V(P)|}{2} = 5 \quad &\leq \omega_f(P; 2) = \chi_f(P; 2) \\ &\leq 2\max(1, \tfrac{3}{2}) + \max(1, \tfrac{4}{2}) = 5, \end{aligned}$$

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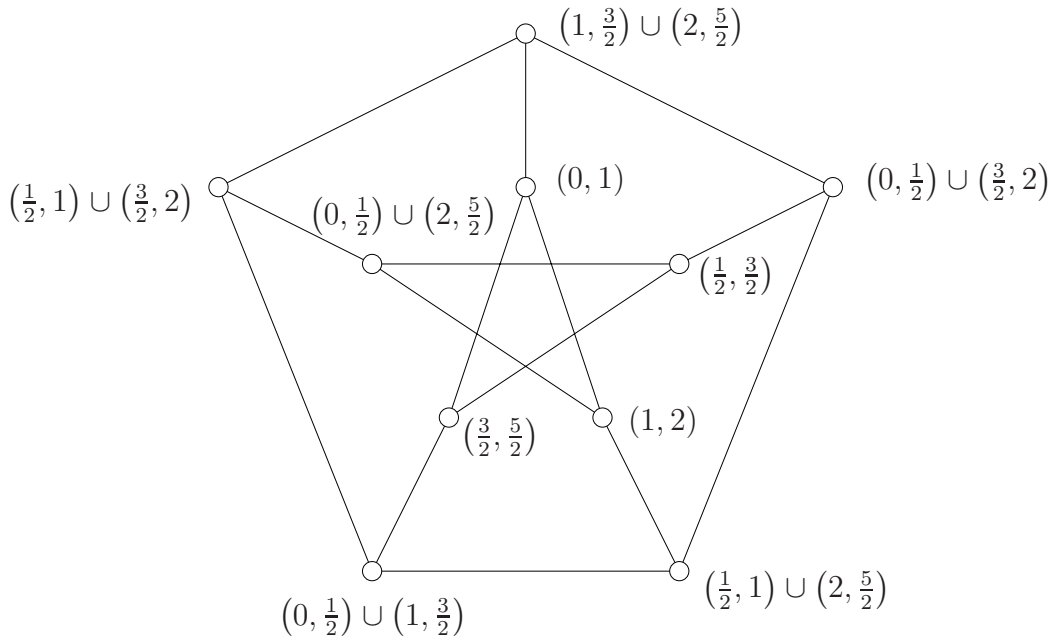

Figure 2: A scheduling function on P with all meetings contained in $(0, \frac{5}{2})$ and each meeting assignment a union of 1 or 2 open intervals

and the same argument with the constant function $1/3$ shows that $\chi_f(P; 3) = 10/3$. When $k = 4 = \alpha(P)$ the upper estimate gives $\chi_f(P) = \chi_f(P; 4) \leq 3$, and here we do not have equality: it can be seen in a number of different ways that $\chi_f(P) = 5/2$. (See below for one of those ways.)

Note that optimal scheduling functions obeying restrictions 2 and 3 on $K_{3,3,4}$ are also optimal scheduling functions obeying the same restriction on P. It follows that in each case there is an optimal scheduling function φ on P such that for each $v \in V(P)$, $\varphi(v)$ is the union of no more than two open intervals. The same holds for $k = 4 = \alpha(P)$, as shown in Figure 2.

Incidentally, the scheduling function shown in Figure 2 shows that $\chi_f(P) \leq \frac{5}{2}$. On the other hand, $\chi_f(P) \geq \frac{|V(P)|}{\alpha(P)} = \frac{10}{4} = \frac{5}{2}$.

We finish this section with the fulfillment of the promise at the end of section 1.

Theorem 3.4. *Suppose that $N, k,$ and $p_1 \leq \cdots \leq p_N$ are positive integers, and $G = K_{p_1} + \cdots + K_{p_N}$. Then $\chi_f(G; k) = \rho(G; k) = \max[p_N, \frac{p_1 + \cdots + p_N}{k}]$.*

Proof. First we shall show that

$$\begin{aligned} \rho(G; k) &= \max\left[\tfrac{|V(H)|}{\alpha(H;k)}; H \text{ is an induced subgraph of } G\right] \\ &= \max\left[p_N, \tfrac{p_1 + \cdots + p_N}{k}\right] \end{aligned}$$

Taking $H = G$, we obtain $\rho(G; k) \geq \frac{p_1 + \cdots + p_N}{k}$. Taking $H = K_{p_N}$, we obtain $\rho(G; k) \geq p_N$. Thus

$$\rho(G; k) \geq \max[p_N, \frac{p_1 + \cdots + p_N}{k}]$$

If H is an induced subgraph of G then $H = K_{s_1} + \cdots + K_{s_t}$ for some $t \geq 1$ and integers $s_1, \ldots s_t$, each no greater than some corresponding p_i. Since $\alpha(H;k) = \min(t,k)$, to maximize $\frac{|V(H)|}{\alpha(H;k)}$ for fixed t we may as well take $s_1, \ldots, s_t$ to be $p_{N-t+1}, \ldots, p_N$. If $t \leq k$ then

$$\frac{|V(H)|}{\alpha(H;k)} \leq \frac{p_{N-t+1} + \cdots + p_N}{t} \leq p_N,$$

and if $k < t$ then

$$\frac{|V(H)|}{\alpha(H;k)} \leq \frac{p_{N-t+1} + \cdots + p_N}{k} \leq \frac{p_1 + \cdots + p_N}{k}.$$

Thus $\rho(G;k) = \max[p_N, \frac{p_1+\cdots+p_N}{k}]$.

Now we will show that $\chi_f(G;k) = \rho(G;k)$. We already have this result when $k \leq N$ and $p_1 = \cdots = p_N$, by Theorem 1.4. We also have this result when $k \geq N$: in this case, clearly $V(G)$ can be partitioned into p_N sets in $\mathcal{I}_N(G) = \mathcal{I}_k(G)$. [This constitutes a proof that $\chi_f(G) = \chi(G) = p_N$.] It is worth noticing that each vertex of G gets scheduled for one continuous unit of time by an optimal scheduling function obeying restriction k, if $N \leq k$.

Fortified by the truth of the theorem when $N \leq k$, when $k \leq N$ and $p_1 = \cdots = p_N$, and also when $k = 1$ ($\chi_f(H;1) = \frac{|V(H)|}{1}$ for all graphs H), we take an arbitrary $k \geq 2$ and show that $\chi_f(G;k) = \rho(G;k)$ for all choices of N and $1 \leq p_1 \leq \cdots \leq p_N$ by induction on $\sum_{i=1}^{N} p_i$. We may as well suppose that $k < N$ and $p_1 < p_N$.

First suppose that $p_N \geq \frac{p_1+\cdots+p_N}{k}$. Then, taking into account that $p_1 < p_N$, p_N appears fewer than k times in the sequence $p_1, \ldots, p_N$. Form $S \in \mathcal{I}_k(G)$, $|S| = k$, by taking one vertex from each of the cliques $K_{p_N-k+1}, \ldots, K_{p_N}$. Then $G - S = K_{q_1} + \cdots + K_{q_M}, q_1 \leq \cdots \leq q_M$, $M \leq N$ ($M < N$ is possible because it could be that $p_{N-k+1} = 1$), with $q_M = p_N - 1$ and $\frac{q_1+\cdots+q_M}{k} = \frac{p_1+\cdots+p_N}{k} - 1 \leq q_M$. By the induction hypothesis, there is a scheduling function on $G - S$ which obeys restriction k, with $\varphi(v) \subseteq (0, p_N - 1)$ for every $v \in V(G-S) = V(G) \backslash S$. Extend φ to $V(G)$ by scheduling every vertex in S for the interval $(p_N - 1, p_N)$, and we have a scheduling function on G which shows that $\chi_f(G;k) \leq p_N = \rho(G;k)$.

Now suppose that $p_N < \frac{p_1+\cdots+p_N}{k}$. If p_N appears no more than k times in the list $p_1, \ldots, p_N$, then an argument similar to that just above works: form $S \in \mathcal{I}_k(G)$ by taking one vertex from each of the k largest K_{p_i} comprising G, apply the induction hypothesis to $G - S$, and wind up with a scheduling function on G which obeys restriction k and schedules all vertices within the interval $(0, \frac{p_1+\cdots+p_N}{k})$. So suppose that p_N occurs at least $k+1$ times on the list $p_1, \ldots, p_N$; i.e. $p_{N-k} = p_N$.

Let $H = NK_1$ be the subgraph of G induced by a subset of $V(G)$ obtained by choosing one vertex from each K_{p_i}, $i = 1, \ldots, N$. Then $G - V(H) = K_{q_1} + \cdots + K_{q_M}$, $1 \leq q_1 \leq \cdots \leq q_M = p_N - 1$; q_M appears in the list $q_1, \ldots q_M$ as many times as p_N does in the list $p_1, \ldots, p_N$, and that number is at least $k+1$. Therefore, $q_M < \frac{q_1+\cdots+q_M}{k}$. Applying the induction hypothesis to $G - V(H)$, and noting that $\frac{q_1+\cdots+q_M}{k} = \frac{p_1+\cdots+p_N}{k} - \frac{N}{k}$, we have a scheduling function φ on $G - V(H)$ which obeys restriction k such that $\varphi(v) \subseteq (0, \frac{p_1+\cdots+p_N}{k} - \frac{N}{k})$ for all $v \in V(G) \backslash V(H)$. By Theorem 1.4, $H = NK_1$ can be properly scheduled with obedience to restriction k with all assignments contained in the interval $\left(\frac{p_1+\cdots+p_N}{k} - \frac{N}{k}, \frac{p_1+\cdots+p_N}{k}\right)$. Extending φ by "concatenation", we see that $\chi_f(G;k) \leq \frac{p_1+\cdots+p_N}{k}$. Therefore, by Corollary 1.1, $\chi_f(G;k) = \rho(G;k)$. $\square$

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Corollary 3.5. (of Theorem 1.4 and the proof of Theorem 3.4) *If $G = K_{p_1} + \cdots + K_{p_N}$ then for every positive integer k there is a scheduling function φ on G which obeys restriction k, with all schedules contained in $(0, \rho(G;k))$, such that for each $v \in V(G)$, $\varphi(v)$ is the union of one or two open intervals.*

4 More chromatic numbers and some problems

Suppose that k and q are positive integers, and G is a graph. We define the *chromatic number* of G with *restriction* k and *allowance* q to be $\chi(G;k,q) = \inf[T$; there is a scheduling function φ on G which obeys restriction k, with $\varphi(v) \subseteq (0,T)$ for every $v \in V(G)$, such that $\varphi(v)$ is the union of no more than q open intervals].

This definition can be regarded as a refinement of a special case of a definition in [1], which is the definition of what we would denote here by $\chi(G;\alpha(G),1)$; that is, in [1] we are scheduling with an allowance 1, but no restriction.

Lemma 4.1 will seem self-evident to many, but there is actually a proof of it in [1].

Lemma 4.1. *For any graph G and integer $k \geq \alpha(G)$, $\chi(G;k,1) = \chi(G)$.*

Lemma 4.2. *Suppose that G is a graph. For any positive integers $k \leq \alpha(G)$ and q*

(a) $\chi(G;k,q) \geq \chi(G;k,q+1)$ and $\chi(G;k,p) = \chi_f(G;k)$ for all $p \geq a(G;k)$, for some (smallest) positive integer $a(G;k)$.

(b) $\chi(G;k,q) \geq \chi(G;k+1,q)$ and $\chi(G;m,q) = \chi(G;\alpha(G),q)$ for all $m \geq b(G;q)$ for some (smallest) integer $b(G;q) \in \{1,\ldots,\alpha(G)\}$.

Problem schema

1. For given G and $k \leq \alpha(G)$, determine $a(G;k)$, as defined in Lemma 4.2(a).
2. For given G and q, determine $b(G;a)$, as defined in Lemma 4.2(b).

Remarks

Regarding Problem scheme 1, Theorem 1.4 says that if $G = NK_p$ and $k \leq \alpha(G) = N$, then $a(G,k) \leq 2$, with equality if and only if $k \nmid Np$.

If $G = K_{p_1} + \cdots + K_{p_N}$, from the proof of Theorem 3.4 we have that $a(G;k) = 1$ if $N \leq k$, and if $k < N$ then $a(G;k) \leq 2$. (The proof involves induction on $p_1 + \cdots + p_N$, as in the proof of Theorem 3.4.) We do not know exactly for which $k < N$ and $p_1, \ldots, p_N$ we have $a(G;k) = 1$.

From the proof of Theorem 3.1 we have that for arbitrary graphs G and H and positive integers k,

$$a(G \vee H;k) = \max[a(G;k), a(H;k)]$$

From this and Theorem 1.4 we have that a $(K_{n_1,\ldots,n_p};k) \leq 2$, with equality if and only if for some $i \in \{1,\ldots,p\}$, $k \leq n_i$ and $k \nmid n_i$.

For the Petersen graph, P, from the discussion in section 3 and Lemma 4.1 we have that $a(P;k) = 2$ for $k \geq 4 = \alpha(P)$. For $k \in \{2,3\}$, we have $a(P;k) \geq a(P;4) = 2$, since every scheduling function obeying restriction k will also obey restriction $k+1$. On the other hand, $a(P;k) \leq a(K_{3,3,4};k) = 2$. Therefore, $a(P;k) = 2, k = 2,3$.

For any graph G, $a(G;1) = 1$. It would be of interest to determine $a(G;\alpha(G))$, the smallest integer q such that G can be properly scheduled within $(0,\chi_f(G))$ with each vertex scheduled for no more than q separate meeting sessions. Call this number $a(G)$, for short. As mentioned above, we have $a(P) = 2$, and from the discussion preceding, $a(K_{p_1}+\cdots+K_{p_n}) = 1$, as well.

Regarding Problem scheme 2, it would be of special interest to find $b(G;1)$ for each G, and also $b(G;a(G;\alpha(G))$, which is the smallest value of k such that $\chi_f(G;k) = \chi_f(G)$. Let us call this number $b(G)$, for short. We would expect that $b(G) = \alpha(G)$, at least for most G, and the results and discussion in section 3 show that this is indeed the case for the Petersen graph and the complete multipartite graphs. But if $G = K_{p_1} + \cdots + K_{p_N}$, $p_1 \leq \cdots \leq p_N$, then Theorem 3.4 shows that $b(G)$ is the smallest k such that $p_N \geq \frac{p_1+\cdots+p_N}{k}$, and this k may well be less than $N = \alpha(G)$.

Appendix

Proof of Theorem 2.2 Let $\chi^*(G;k)$ denote the minimum on the right hand side of the equation above. By previous remarks, in minimizing the sum $\sum_{S\in\mathcal{I}_k} f(S)$ we may as well confine our attention to fractional proper colorings $f : \mathcal{I}_k(G) \to [0,1]$, which form a closed subset of the compact set $[0,1]^{|\mathcal{I}_k|}$. Therefore the minimum in the definition of $\chi^*(G;k)$ exists. Suppose that f is a fractional proper coloring of G satisfying restriction k such that $\sum_{S\in\mathcal{I}_k(G)} f(S) = \chi^*(G;k)$. Let the sets S in $\mathcal{I}_k$ for which $f(S) > 0$ be $S_1,\ldots,S_q$. Let

$$I_1 = (0, f(S_1)), I_2 = (f(S_1), f(S_1)+f(S_2)), \ldots, I_q = (\sum_{j<q} f(S_j), \sum_{j\leq q} f(S_j)).$$

Define g on $V(G)$ by

$$g(v) = \cup_{\{j|v\in S_j\}} I_j.$$

Then for each $v \in V(G)$, $g(v)$ is a union of open intervals and

$$g(v) \subseteq (0, \sum_{j=1}^{q} f(S_j)) = (0, \chi^*(G,k)).$$

Since the intervals $I_1,\ldots,I_q$ are pairise disjoiont, of lengths $f(S_1),\ldots,f(S_q)$, respectively,

$$\begin{array}{ll} \mu(g(v)) = & \sum_{\{j|v\in S_j\}} \mu(I_j) \\ = & \sum_{\{j|v\in S_j\}} f(S_j) = \sum_{\{S\in\mathcal{I}_k|v\in S\}} f(S) \geq 1, \end{array}$$

since the S_j are the only sets in $\mathcal{I}_k$ at which $f > 0$.

Since the I_j are pairwise disjoint, $t \in \mathbb{R}$ can lie in at most one of them, and since $S_j \in \mathcal{I}_k(G)$, if $t \in I_j$, then $|\{v \in V(G)|t \in g(v)\}| = |S_j| \leq k$. That is, g obeys restriction k.

If $uv \in E(G)$ then u and v cannot both lie in the same S_j, so $g(u) \cap g(v) = \emptyset$.

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By Lemma 2.1 there is a scheduling function $\tilde{g}$ on G such that for each $v \in V(G)$, $\tilde{g}(v) \subseteq g(v) \subseteq (0, \chi^*(G;k))$; since such a $\tilde{g}$ must obey restriction k, it follows that

$$\chi_f(G;k) \leq \chi^*(G;k).$$

Now suppose that $\epsilon > 0$ and that φ is a scheduling function on G obeying restriction k such that $\varphi(v) \subseteq (0, \chi_f(G;k) + \epsilon)$ for all $v \in V(G)$. Let $I_1, \ldots, I_p$ be open subintervals of $(0, \chi(G;k) + \epsilon)$ such that each set $\varphi(v)$, $v \in V(G)$, is a union of some of the I_j, and let $J_1, \ldots, J_q$ be pairwise disjoint open intervals such that each J_i is contained in some I_j, and each I_j is, except possibly for finitely many points, a union of some of the J_i. Define g on $V(G)$ by $g(v) = \bigcup_{\{r|J_r \subseteq \varphi(v)\}} J_r$. Then for each $v \in V(G)$, $g(v) \subseteq \varphi(v) \subseteq (0, \chi_f(G;k) + \epsilon)$ and $\varphi(v) \backslash g(v)$ is finite. Clearly g is a scheduling function on G obeying restriction k, because φ is such a function. Note that for each $v \in V(G)$ and each r, either $J_r \subseteq g(v)$ or $g(v) \cap J_r = \emptyset$.

For $r = \{1, \ldots, q\}$, let $S_r = \{v \in V(G) \mid J_r \subseteq g(v)\}$. Because g is a scheduling function on G which obeys restriction k, $S_r \in \mathcal{I}_k(G)$. Define $f : \mathcal{I}_k(G) \to [0, \infty)$ by $f(S_r) = \mu(J_r)$, $r = 1, \ldots, q$, and $f(S) = 0$ for $S \in \mathcal{I}_k \backslash \{S_1, \ldots, S_q\}$. Because the J_r are pairwise disjoint, we have that for each $v \in V(G)$,

$$\begin{aligned} \textstyle\sum_{\{S \in \mathcal{I}_k | v \in S\}} f(S) &= \textstyle\sum_{\{r|v \in S_r\}} f(S_r) = \sum_{\{r|J_r \subseteq g(v)\}} \mu(J_r) \\ &= \mu(\textstyle\bigcup_{\{r|J_r \subseteq g(v)\}} J_r) = \mu(g(v)) = 1, \end{aligned}$$

since g is a scheduling function on G. Therefore, f is a fractional proper coloring of G satisfying restriction k; consequently

$$\begin{aligned} \chi^*(G;k) &\leq \textstyle\sum_{S \in \mathcal{I}_k} f(S) = \sum_{r=1}^{q} f(S_r) \\ &= \textstyle\sum_{r=1}^{q} \mu(J_r) = \mu(\bigcup_{r=1}^{q} J_r) \\ &\leq \mu((0, \chi_f(G;k) + \epsilon)) = \chi_f(G;k) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\chi^*(G;k) \leq \chi_f(G;k)$. $\square$

Proof of Corollary 2.3 In the first part of the proof of Theorem 2.2 the fact that $\chi^*(G;k)$ is a minimum is used to produce a scheduling function $\tilde{g}$ on G obeying restriction k such that $\tilde{g}(v) \subseteq (0, \chi^*(G;k))$ for each $v \in V(G)$. Since $\chi^*(G;k) = \chi_f(G;k)$, this shows that the infimum that $\chi_f(G;k)$ is defined to be is actually achieved—i.e., is a minimum.

The other conclusions of this corollary are straightforward from the theory of *linear programming*, for $\chi^*(G;k)$ is the *value* of the linear program: minimize $\sum_{S \in \mathcal{I}_k(G)} f(S)$ subject to the constraints

$$\begin{gathered} f(S) \geq 0 \text{ for all } S \in \mathcal{I}_k(G) \\ \text{and } Mf \geq \mathbf{1}, \end{gathered}$$

in which $\mathbf{1}$ is column vector of $|V(G)|$ ones, f is a column vector of the values $f(S)$, $S \in \mathcal{I}_k(G)$, in some order, and M is, with respect to the same ordering on $\mathcal{I}_k$, a $V(G) - \mathcal{I}_k(G)$ *incidence matrix*, with rows indexed by the vertices of G, columns by the sets $S \in \mathcal{I}_k(G)$, with entries $M(v,S) = \{ \begin{array}{ll} 0 & \text{if } v \notin S \\ 1 & \text{if } v \in S. \end{array}$ The inequality $Mf \geq \mathbf{1}$ means that each entry of Mf is at least 1.

Because there is some solution, and the entries of M are rational, there is a rational solution f. Thus $\chi^*(G;k) = \chi_f(G;k) = \sum_{S\in\mathcal{I}_k(G)} f(S)$ is rational, and the pairwise disjoint intervals $I_1,\ldots,I_q$ used in the first half of the proof of Theorem 2.2 to define the function g have rational endpoints. Since $g(v)$ is a union of some of these intervals it follows that $\mu(g(v))$ is rational, for each $v \in V(G)$. Therefore, in the adjustment of g to a scheduling function $\tilde{g}$, by a shrinking of some of the intervals assigned by g to v when $\mu(g(v)) > 1$, as in Lemma 2.1, so that $\mu(\tilde{g}(v)) = 1$, since the sum of the lengths of the intervals to be shrunk is rational, and their endpoints are rational, and the amount by which the sum of their lengths is to be reduced, $\mu(g(v)) - 1$, is rational, arrangements can be made so that the intervals whose union is $\tilde{g}(v)$ have rational endpoints. □

Proof of Corollary 2.4 It is straightforward to see that $\omega_f(G;k)$ is the value of a linear program which is the dual of the one for $\chi^*(G;k)$, in the proof of Theorem 2.2. By the Duality Theorem of linear programming (see [2], Appendix 3), therefore, $\omega_f(G;k) = \chi^*(G;k) = \chi_f(G;k)$. □

Proof of Theorem 2.5 We will use the characterization of $\chi_f(G;k)$ given by Theorem 2.2, as well as Corollary 2.3.

Suppose that $f : \mathcal{I}_k(G) \to [0,1]$ is a rational-valued fractional proper coloring of G satisfying restriction k, such that

$$\chi_f(G;k) = \sum_{S\in\mathcal{I}_k(G)} f(S).$$

Let r be a common denominator of all the values $f(S)$, $S \in \mathcal{I}_k(G)$, and set

$$\chi_f(G;k) = \frac{m}{r} = \sum_{j=1}^{t} f(S_j),$$

where $\{S_1,\ldots,S_t\} = \{S \in \mathcal{I}_k(G) \mid f(S) > 0\}$. Let $f(S_i) = \frac{n_i}{r}$, so that $\sum_{i=1}^{t} n_i = m$. Let $J_q = \{\sum_{i<q} n_i + 1, \ldots, \sum_{i\le q} n_i\}$, $q = 1,\ldots,t$. For each $v \in V(G)$, let $g(v) = \bigcup_{\{q|v\in S_q\}} J_q$; since $\sum_{\{S\in\mathcal{I}_k(G)|v\in S\}} f(S) \ge 1$, it follows that $|g(v)| \ge r$. Define $\varphi : V(G) \to \binom{[m]}{r}$ by $\varphi(v) =$ some r-subset of $g(v)$.

Since $uv \in E(G)$ implies that u and v cannot belong to the same S_q and the J_q are pairwise disjoint, it follows that if $uv \in E(G)$ then $\varphi(u) \cap \varphi(v) \subseteq g(u) \cap g(v) = \emptyset$; and since f satisfies restriction k (each S_j has no more than k elements), φ satisfies the following: for each $j \in \{1,\ldots,m\}$,

$$|\{v \in V(G) \mid j \in \varphi(v)\}| \le |\{v \mid j \in g(v)\}| = |S_q| \le k$$

for the value of q such that $j \in J_q$.

Thus $m \ge \chi^{(r)}(G;k)$, so

$$\begin{aligned}\chi_f(G;k) = \tfrac{m}{r} &\ge \tfrac{1}{r}\chi^{(r)}(G;k)\\ &\ge \inf_p \tfrac{1}{p}\chi^{(p)}(G;k).\end{aligned}$$

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Notice that the proof so far allows us to conclude that if $\chi_f(G;k) \leq \inf_p \frac{1}{p}\chi^{(p)}(G;k)$, then $\chi_f(G;k) = \inf_p \frac{1}{p}\chi^{(p)}(G;k) = \min_p \frac{1}{p}\chi^{(p)}(G;k)$.

Now suppose that $\epsilon > 0$, $\frac{1}{r}\chi^{(r)}(G;k) < \inf_p \frac{1}{p}\chi^{(p)}(G;k) + \epsilon$, $m = \chi^{(r)}(G;k)$, and $\varphi : V(G) \to \binom{[m]}{r}$ is an (m,r)-coloring of G which satisfies restriction k. For each $q \in \{1,\ldots,m\}$, let $S_q = \{v \in V(G) \mid q \in \varphi(v)\}$. Then $S_q \in \mathcal{I}_k(G)$; also, each $v \in V(G)$ belongs to S_q for r different values of q—namely, for each $q \in \varphi(v)$.

Define $f : \mathcal{I}_k(G) \to [0,1]$ by $f(S_q) = 1/r$, $q = 1,\ldots,m$ and $f(S) = 0$ for $S \in \mathcal{I}_k(G)\backslash\{S_1,\ldots,S_m\}$. (If some $S \in \mathcal{I}_k(G)$ should appear more than once—say t times, $t > 1$—in the list $S_1,\ldots,S_m$— then the intention of the preceding definition of f is that $f(S) = \frac{t}{r}$.) Then for each $v \in V(G)$, $\sum_{\{S\in\mathcal{I}_k(G)|v\in S\}} f(S) = \frac{r}{r} = 1$. Thus f is a fractional proper coloring of G satisfying restriction k, so, by Theorem 2.2,

$$\begin{aligned}\chi_f(G;k) &\leq \textstyle\sum_{S\in\mathcal{I}_k(G)} f(S) = \frac{m}{r}\\ &= \tfrac{1}{r}\chi^{(r)}(G;k) < \inf_p \tfrac{1}{p}\chi^{(p)}(G;k) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that

$$\chi_f(G;k) \;\leq\; \inf_p \frac{1}{p}\chi^{(p)}(G;k).$$

Thus, by previous remarks,

$$\chi_f(G;k) = \min_p \frac{1}{p}\chi^{(p)}(G;k).$$

It is straightforward to see that for each G and k, $r \to \chi^{(r)}(G;k)$ is subadditive: for positive integers p, q,

$$\chi^{(p+q)}(G;k) \leq \chi^{(p)}(G;k) + \chi^{(q)}(G;k).$$

Therefore, by [2], Appendix 4,

$$\begin{aligned}\chi_f(G;k) &= \inf_r \tfrac{1}{r}\chi^{(r)}(G;k)\\ &= \lim_{r\to\infty} \tfrac{1}{r}\chi^{(r)}(G;k).\end{aligned}$$

□

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