A Partition Bijection Relating the Rogers-Selberg Identities to a Special Case of Gordon’s Theorem

Andrew V. Sills
Georgia Southern University, asills@georgiasouthern.edu

Follow this and additional works at: https://digitalcommons.georgiasouthern.edu/math-sci-facpres

Part of the Mathematics Commons

Recommended Citation

This presentation is brought to you for free and open access by the Mathematical Sciences, Department of at Digital Commons@Georgia Southern. It has been accepted for inclusion in Mathematical Sciences Faculty Presentations by an authorized administrator of Digital Commons@Georgia Southern. For more information, please contact digitalcommons@georgiasouthern.edu.
A Partition Bijection Related to the Rogers-Selberg Identities

Andrew Sills
Rutgers University

Québec-Maine Number Theory Conference
October 1, 2006
The Rogers-Ramanujan Identities.

\[ \sum_{j=0}^{\infty} \frac{q^j}{q^{j^2} + 1} = \prod_{j=1 \atop j \equiv \pm 1 \mod 5}^{j \neq 0, \pm 1 \mod 5} (1 - q^j) \]

and

\[ \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1 - q)(1 - q^2) \cdots (1 - q^j) (b - 1)} = \prod_{j=1 \atop j \equiv \pm 2 \mod 5}^{j \neq 0, \pm 2 \mod 5} (1 - q^j) . \]
Assume throughout that $|q| < 1$.

Rising $q$-factorial notation

$$(a; q)_n := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$$

$$(a; q)_\infty := (1 - a)(1 - aq)(1 - aq^2) \cdots$$
The Rogers-Ramanujan Identities.

\[ \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q; q)_j} = \prod_{j \geq 1, j \neq 0, \pm 1 \pmod{5}} \frac{1}{1 - q^j} \]

and

\[ \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q; q)_j} = \prod_{j \geq 1, j \neq 0, \pm 2 \pmod{5}} \frac{1}{1 - q^j} \]
The Rogers-Selberg Identities.

\[
\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+2}; q)_{\infty}}{(q^2; q^2)_j} = \prod_{j \geq 1} \frac{1}{1 - q^j}, \\
j \not\equiv 0, \pm 1 \pmod{7}
\]

\[
\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+1}; q)_{\infty}}{(q^2; q^2)_j} = \prod_{j \geq 1} \frac{1}{1 - q^j}, \\
j \not\equiv 0, \pm 2 \pmod{7}
\]

and

\[
\sum_{j=0}^{\infty} \frac{q^{2j^2}(-q^{2j+1}; q)_{\infty}}{(q^2; q^2)_j} = \prod_{j \geq 1} \frac{1}{1 - q^j}, \\
j \not\equiv 0, \pm 3 \pmod{7}
\]
A partition $\pi$ of an integer $n$ is a nonincreasing finite sequence of positive integers

$$\pi = \{\pi_1, \pi_2, \pi_3, \ldots, \pi_s\}$$

such that $\sum_i \pi_i = n$.

Each nonzero term in $\{\pi_1, \pi_2, \pi_3, \ldots, \pi_s\}$ is called a part of the partition $\pi$.

The seven partitions of 5 are thus

$$\{5\} \quad \{4, 1\} \quad \{3, 2\} \quad \{3, 1, 1\} \quad \{2, 2, 1\} \quad \{2, 1, 1, 1\} \quad \{1, 1, 1, 1, 1\}.$$
The multiplicity of the integer $j$ in the partition $\pi$, denoted $m_j(\pi)$, is the number of times $j$ appears in $\pi$.

$$\pi = \langle 1^{m_1(\pi)} 2^{m_2(\pi)} 3^{m_3(\pi)} \ldots \rangle$$

The seven partitions of 5 are thus

$$\langle 5 \rangle \quad \langle 1 \ 4 \rangle \quad \langle 2 \ 3 \rangle \quad \langle 1^2 \ 3 \rangle \quad \langle 1 \ 2^2 \rangle \quad \langle 1^3 \ 2 \rangle \quad \langle 1^5 \rangle.$$
The Rogers-Ramanujan Identities—Combinatorial Version. For \( i = 1, 2, \)
the number of partitions of \( n \) into parts which are nonconsecutive integers greater than \( 2 - i \)
and in which no part is repeated
equals
the number of partitions of \( n \) into parts \( \not\equiv 0, \pm i \) (mod 5).
Gordon's Theorem.

Let $G_{k,i}(n)$ denote the number of partitions of $n$ into parts such that 1 appears as a part at most $i - 1$ times and the total number of appearances of any two consecutive integers is at most $k - 1$.

Let $C_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k + 1}$.

Then $G_{k,i}(n) = C_{k,i}(n)$ for $1 \leq i \leq k$ and all integers $n$. 
Theorem 1 (Andrews).

Let $A_2(n)$ denote the number of partitions of $n$ such that if $2j$ is the largest repeated even part, then all positive even integers less than $2j$ also appear at least twice, no odd part less than $2j$ appears, and no part greater than $2j$ is repeated.

Then $A_2(n) = C_{3,2}(n)$ for all $n$. 
Let $G_2$ denote the set of partitions enumerated by $G_{3,2}(n)$ in Gordon’s theorem, i.e. partitions $\pi$ such that

$$m_1(\pi) \leq 1$$

and

$$m_j(\pi) + m_{j+1}(\pi) \leq 2$$

for all $j \geq 1$.

Let $A_2$ denote the set of partitions enumerated by $A_2(n)$ in Theorem 1, i.e. partitions $\pi$ such that

$$m_j(\pi) \leq 1 \text{ if } j \text{ is odd},$$

$$m_j(\pi) = 0 \text{ if } j \text{ is odd and } j < R(\pi),$$

$$m_j(\pi) \geq 2 \text{ if } j \text{ is even and } j < R(\pi),$$

where $R(\pi)$ is the largest repeated part in $\pi$. 
A partition $\pi \in G_2$ is one in which

- no number appears more than twice as a part,
- if $r$ appears twice, then neither $r - 1$ nor $r + 1$ appear, and
- $1$ appears at most once.
A partition $\pi \in \mathcal{A}_2$ may be thought of as a union of two partitions:

- a partition into 2’s, 4’s, 6’s, . . . , $2^j$’s with all parts repeated, and

- a partition into distinct parts greater than $2^j$. 
Example.

\[
\{30, 27, 19, 15, 15, 13, 12, 10, 8, 8, 4, 4, 2, 1\} \in G_2 \\
\downarrow \\
\{30, 27, 19, 15, 14, 12, 8, 7\} \cup \langle 2^4 4^4 6^2 \rangle \in A_2
\]
<table>
<thead>
<tr>
<th>18</th>
<th>17</th>
<th>16</th>
<th>15</th>
<th>14</th>
<th>13</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>27</td>
<td>19</td>
<td>15</td>
<td>14</td>
<td>12</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

So first row is

30 27 19 2 2 2 2 2 2 2 2 2 2 2 2 2
So first row and second rows are

```
  11 10 9 8 7 6 5 4 3 2 1
  13 12 10 6 5 4 4 2 2 2 2
```

↑

```
30 27 19 2 2 2 2 2 2 2 2 2 2 2 2 2 2
13 12 10 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
```
So the first three rows are

\[
\begin{array}{ccccccc}
4 & 3 & 2 & 1 \\
4 & 3 & 2 & 2 \\
\uparrow \\
\end{array}
\]

30 & 27 & 19 & 2 & 2 & 2 & 2 \\
13 & 12 & 10 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2
\[
\begin{array}{c}
\frac{2}{2} \\
\frac{1}{1} \\
\uparrow \\
+ & 13 & 12 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
+ & 2 & 2 & 2 & 2 \\
+ & 2 & 1 \\
\hline
\end{array}
\]
Analogous interpretations of the other two Rogers-Selberg identities can be given and they in turn can be mapped similarly to the $i = 1$ and $i = 3$ instances of the partitions enumerated by the $k = 3$ case of Gordon’s theorem.