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Rainbow Generalizations of Ramsey Theory - A Dynamic Survey

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Abstract

In this work, we collect Ramsey-type results concerning rainbow edge colorings of graphs.

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If you have corrections, updates or new results which fit the scope of this work, please contact Yaping Mao at yapingmao@outlook.com or maoyaping@ymail.com.

Contents

1 Introduction

We will almost entirely focus on coloring edges so "coloring" will mean edge coloring. In most cases, k will be used to denote the number of colors used on the edges. Also define the color degree $d^c(v)$ to be the number of colors on edges incident to v. A colored graph is called *rainbow* if each edge receives a distinct color. For all other notation, we refer the reader to [49].

The original publication of this work was [89]. There are some other surveys of edge coloring that we should mention. The first is the dynamic survey [199] by Radziszowski which contains a wonderful list of known (monochromatic) Ramsey numbers. There is a brief survey of anti-Ramsey results in [207]. Also there is a survey by Kano and Li [140] which discusses some rainbow coloring. There is also a forthcoming survey by Fujita, Liu and Magnant [85] related to this survey but focusing more on large monochromatic structures.

It should be noted that in [216], Voloshin demonstrates very interesting relationships between rainbow / monochromatic subgraphs and mixed hypergraph colorings. In fact, many of the notions of generalized Ramsey colorings are very closely related to upper and lower chromatic numbers of the derived mixed hypergraph.

2 Anti-Ramsey Theory

The anti-Ramsey problem is stated as follows.

Definition 1 *Given graphs G and H with $G \supset H$, the* anti-Ramsey number $ar(G, H)$ *is defined to be the maximum number of colors k such that there exists a coloring of the edges of G with exactly k colors in which every copy of H in G has at least two edges with the same color (H is not rainbow colored).*

Classically, the graph G is a large complete graph and the graph H comes from some class.

This is equivalent to the *rainbow number* $rb(G, H)$ which is defined to be the minimum number of colors k such that any coloring, using k colors, of the edges of G contains a rainbow H. Thus, the relationship is $rb(G, H) = \mathrm{ar}(G, H) + 1$. In order to be consistent with the majority of the results, we state all results in terms of anti-Ramsey numbers.

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The study of anti-Ramsey theory began with a paper by Erdős, Simonovits, and Sós [67] in 1975 (note that related ideas were studied even earlier in [66]). Since then, the field has blossomed in a wide variety of papers. See [207] for a brief survey.

2.1 Cycles

In the original work by Erdős, Simonovits and Sós, the authors stated the following conjecture.

Conjecture 1 ([67]) *For all $n \geq k \geq 3$,*

$$ar(K_n, C_k) = \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1).$$

The authors provided the following lower bound construction (as presented in [128]). For $n = (k-1)q + r$, partition $V(G)$ into sets $V_1, \dots, V_q$ of size $k-1$ and one set V_{q+1} of size r. The edges with endpoints in the same set receive $q\binom{k-1}{2} + \binom{r}{2}$ different colors. On the remaining edges, we use q more colors $c_1, \dots, c_q$ with $c_{\min\{i,j\}}$ on the edges between the sets V_i and V_j when $i \neq j$. Each set is too small to contain the desired cycle and there can be no cycle between sets so this example provides the stated lower bound.

Erdős, Simonovits and Sós proved the conjecture in the case when $k = 3$ by showing that $\mathrm{ar}(K_n, C_3) = n - 1$. Alon [3] proved the conjecture for $k = 4$ by showing that $\mathrm{ar}(K_n, C_4) = \left\lfloor \frac{4n}{3} \right\rfloor - 1$. He also provided a general upper bound of $\mathrm{ar}(K_n, C_k) \leq (k-2)n - \binom{k-1}{2}$. In 2000, Montellano-Ballesteros and Neumann-Lara [189] provided another upper bound. Jiang and West [128] later improved the upper bound to $\mathrm{ar}(K_n, C_k) \leq \left(\frac{k+1}{2} - \frac{2}{k-1}\right) n - (k-2)$ with a slight improvement when k is even. In 2004, Jiang, Schiermeyer and West [126] (see also [205]) proved the conjecture for $k \leq 7$ but finally, in 2005, Montellano-Ballesteros and Neumann-Lara [192] completely proved Conjecture 1 with a simplified proof by Choi in [55].

Theorem 2.1 ([192]) *For all $n \geq k \geq 3$,*

$$ar(K_n, C_k) = \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1).$$

In a related work, Axenovich, Jiang and Kündgen [18], proved the following result for finding even cycles in complete bipartite graphs.

Theorem 2.2 ([18]) *For all positive integers m, n, k with $m \leq n$ and $k \geq 2$,*

$$ar(K_{m,n}, C_{2k}) = \begin{cases} (k-1)(m+n) - 2(k-1)^2 + 1 & \text{for } m \geq 2k-1, \\ (k-1)n + m - (k-1) & \text{for } k-1 \leq m \leq 2k-1, \\ mn & \text{for } m \leq k-1. \end{cases}$$

For the general class of all rainbow cycles, the following was shown.

Theorem 2.3 ([134]) *For positive integers m and n, the maximum number of colors that can appear in an edge coloring of $K_{m,n}$ with no rainbow cycles is $m + n - 1$.*

It was also shown in [134] that the colorings that achieve the bound in Theorem 2.3 can be encoded by special vertex labelings of full binary trees with $m+n$ leaves.

Looking within hypercubes, the authors of [36] consider cycles and provide some bounds on $\mathrm{ar}(Q_n, C_k)$ and the exact results when $n \leq 4$.

Let Ω_k be the set of graphs containing k vertex disjoint disjoint cycles. In [132], the following result was proven along with some general bounds for $ar(K_n, \Omega_k)$.

Theorem 2.4 ([132]) *For $n \geq 7$,*

$$ar(K_n, \Omega_2) = 2n - 2.$$

Also $ar(K_6, \Omega_2) = 11$.

Gorgol [95] considered using a split graph as the underlying host graph.

Theorem 2.5 ([95]) *Let H be a graph with $\delta(G) \geq 2$. Then*

$$ar(K_n + \overline{K_s}, H) \geq ar(K_n, H) + s$$

for $n, s \geq 1$.

Theorem 2.6 ([95]) *If $|V(H)| \leq n$ and H is a subgraph of $K_{n,s}$, then*

$$ar(K_n + \overline{K_s}, H) \geq ar(K_n, H) + ar(K_{n,s}, H).$$

Theorem 2.7 ([95]) *Let $n \geq 2$ and $s \geq 1$. Then $ar(K_n + \overline{K_s}, C_3) = n + s - 1$.*

Proposition 1 ([95]) *Let $n \in \{2, 3\}$ and $n + s \geq 4$. Then*

$$ar(K_n + \overline{K_s}, C_4) = ar(K_{n,s}, C_4) + 1.$$

Theorem 2.8 ([95]) *Let $n \geq 4$ and $s \geq n$. Then*

$$ar(K_n + \overline{K_s}, C_4) \leq ar(K_n, C_4) + ar(K_{n,s}, C_4) - 1.$$

Letting C_3^+ denote a triangle with a pendant edge, the following was obtained.

Theorem 2.9 ([95]) *Let $n \geq 3$ and $s \geq 1$. Then*

$$ar(K_n + \overline{K_s}, C_3^+) \leq n + s - 1.$$

Let B be the bull, the triangle with two disjoint pendant edges.

Theorem 2.10 ([95]) *Let $n, s \geq 1$ and $n + s \geq 5$. Then*

$$ar(K_n + \overline{K_s}, B) \geq n + s,$$

and this bound is sharp for $n = 2, 3$.

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Let $K_{1,4}^+$ denote the triangle with two pendant edges incident to a single vertex of the triangle.

Theorem 2.11 ([95]) *Let $s \geq 3$. Then*

$$ar(K_2 + \overline{K_s}, K_{1,4}^+) \leq s + 1,$$
$$ar(K_3 + \overline{K_s}, K_{1,4}^+) \leq \max\{7, s + 3\}.$$

Theorem 2.12 ([95]) *Let $n \geq 4$ and $s \leq n$. Then*

$$ar(K_n + \overline{K_s}, K_{1,4}^+) \leq n + s + 1.$$

Let W_d be a wheel obtained from C_d by adding a new vertex v, the central vertex of W_d, and joining v to all vertices of C_d.

Xu, Lu and Liu derived the following results.

Theorem 2.13 ([226]) (1) *For $d \geq 4$, $ar(W_d, C_3) = d + 1$ and $ar(W_d, C_4) = \lfloor \frac{4}{3}d \rfloor$.*

(2) *For any integer $k \geq 6$ and $d \geq k - 1$, we have $ar(W_d, C_k) = \lfloor \frac{2k-7}{k-3}d \rfloor$.*

Theorem 2.14 ([226]) (1) *For $m, n \geq 2$, we have*

$$ar(P_m \square P_n, C_4) = mn + \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - 1.$$

(2) *Let $m \geq 2$ be an integer. Then*

$$ar(P_m \square C_n, C_4) = \begin{cases} 4m - 1 & if\ n = 3; \\ mn + \frac{m-1}{2} \lfloor \frac{n}{2} \rfloor & if\ n \geq 5 \text{ and } m \text{ is odd}, \\ mn + \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1 & if\ n \geq 5 \text{ and } m \text{ is even}. \end{cases}$$

Moreover, $ar(P_m \square C_4, C_4) = 5m - 2$.

Theorem 2.15 ([226]) *Let m, n be two integers and $m \geq n$. Then*

$$ar(C_m \square C_n, C_4) = \begin{cases} 4m & if\ n = 3,\ m \neq 4, \\ mn + \frac{mn}{4} & if\ n \geq 5 \text{ and } m \text{ is even}, \\ mn + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1 & if\ n \geq 5 \text{ and } m \text{ is odd}. \end{cases}$$

A cyclic Cayley graph $Cay(\mathbb{Z}_n, k)$, where $V(Cay(\mathbb{Z}_n, k)) = \mathbb{Z}_n = \{0, 1, ..., n-1\}$ and $(i, j) \in E(Cay(\mathbb{Z}_n, k))$ if and only if $j - i \in \{\pm 1, \pm 2, ..., \pm k\}$. Then $Cay(\mathbb{Z}_n, k)$ is a $2k$-regular graph.

Theorem 2.16 ([226]) *Let $n \geq 6$ be integer. Then $ar(Cay(\mathbb{Z}_n, 2), 3) = n + 1$ if $n \geq 7$ and $ar(Cay(\mathbb{Z}_6, 2), 3) = 6$.*

2.2 Cliques

Let $\mathrm{ex}(n, \mathscr{H})$ be the maximum number of edges in a graph G on n vertices containing no subgraph isomorphic to $H \in \mathscr{H}$. This function has been called the Turán function since it was first studied in [214] where the set $\mathscr{H}$ consists of a single clique of order $k+1$. In this work, Turán proved the following theorem.

Theorem 2.17 ([214])

$$ex(n, K_{k+1}) = \binom{k}{2}t^2 + i(k-1)t + \binom{i}{2}$$

where the sharpness is given by the complete k*-partite graph with partite sets* $V_1, \ldots, V_k$ *where* $|V_j| = t+1$ *for* $1 \leq j \leq i$ *and* $|V_j| = t$ *for* $i+1 \leq j \leq k$ *where* $n = tk + i$ *(i.e. an almost balanced complete multipartite graph).*

On the surface, Theorem 2.17 may seem to have little in common with anti-Ramsey theory but in [67], Erdős, Simonovits, and Sós proved the following relationship.

Theorem 2.18 ([67]) *Given an integer* k*, there exists an integer* $n(k)$ *such that*

$$ar(K_n, K_k) = ex(n, K_{k-1}) + 1 \tag{1}$$

for all $n \geq n(k)$*.*

The authors also showed, in [67], that Equation (1) holds for $k = 3$ for all $n \geq 4$.

Independently, Montellano-Ballesteros and Neumann-Lara [190] and Schiermeyer [206] proved tha Equation (1) holds for all $n > k \geq 3$.

The lower bound, as observed in [67], uses a different color on each edge of Turán's construction and then a single new color on all other edges to complete the coloring. This coloring certainly has no rainbow K_k but it uses $\mathrm{ex}(n, K_{k-1}) + 1$ colors. Both proofs of the upper bound are by induction on n but each uses a different counting strategy within the induction.

The idea of anti-Ramsey numbers for cliques was extended in [34] to coloring in rounds. For positive integers $k \leq n$ and t, let $\chi^t(k, n)$ denote the minimum number χ of colors such that there exists a sequence of length t of χ-colorings $\psi_1, \psi_2, \ldots, \psi_t$ of the edges of K_n such that all $\binom{k}{2}$ edges of each $K_k \subseteq K_n$ get different colors in at least one coloring ψ_i. Conversely, let $t(k, n)$ denote the minimum length of such a sequence of colorings each using $\binom{k}{2}$ colors such that each K_k is rainbow in at least one coloring. The main result of [34] is the following concerning rainbow triangles.

Theorem 2.19 ([34]) *For all* $n \geq 3$ *and* t*,*

$$(n-1)^{1/t} \leq \chi^t(3, n) \leq 4n^{1/t} - 1.$$

This result generalizes an earlier result of Körner and Simonyi [145] which is stated as follows.

Theorem 2.20 ([145]) *For all $n \geq 3$,*

$$\lceil \log(n-1)/\log 3 \rceil \leq t(3,n) \leq \lceil \log n \rceil .$$

The authors of [34], then go on to explore 2-round colorings, providing bounds on $\chi^2(i,n)$ for $i = 4,5,6,7$. Further, they studied $t(k,n)$ for $k = n-1, n-2$ and $\frac{n}{2}$.

2.3 Trees

For general trees, Jiang and West [127] provide exact numbers for some families of trees and bounds for some individual trees. Let $\mathscr{T}_k$ be the family of all trees on k edges and let $\ell(n,k)$ denote the maximum size of an n-vertex graph in which every two components together have at most k vertices.

For the sake of notation, for any set of graphs $\mathscr{H}$, let $\text{ar}(K_n, \mathscr{H})$ be the maximum number of colors k such that there exists a coloring of K_n with exactly k colors in which, for all $H \in \mathscr{H}$, no copy of H in the colored K_n is rainbow.

Theorem 2.21 ([127]) *If $n > k$, then:*

$$ar(K_n, \mathscr{T}_k) - 1 = \ell(n,k) = \begin{cases} \binom{k-1}{2} & \text{if } k < n \leq 2k-1, \\ \binom{\lceil k/2 \rceil}{2} + r\binom{\lfloor k/2 \rfloor}{2} + \binom{s}{2} & \text{if } n \geq 2k. \end{cases}$$

where $r = \lfloor (n - \lceil k/2 \rceil)/\lfloor k/2 \rfloor \rfloor$ and $s = n - \lceil k/2 \rceil - r\lfloor k/2 \rfloor$.

This result is proven by finding $\ell(k,n)$ and then showing the relationship to the anti-Ramsey number. The bipartite version of this problem is considered in [131]. Also in [127], the authors prove the following for an individual tree T.

Theorem 2.22 ([127]) *Let T be a tree with k edges and $n \geq 2k$. Then*

$$\frac{n}{2}\left\lfloor \frac{k-2}{2} \right\rfloor + c_k \leq ar(K_n, T) \leq n(k-1)$$

where c_k does not depend on n.

The upper bound in Theorem 2.22 comes from the known bound of $\text{ex}(n,T) \leq n(k-1)$. Regarding this quantity, Erdős and Sós conjectured the following.

Conjecture 2

$$ex(n,T) \leq \frac{n(k-1)}{2}.$$

If this conjecture is true, then the upper bound of Theorem 2.22 can also be reduced to $\frac{n(k-1)}{2}$.

More specifically, Jiang and West also proved the following result for brooms. Let $B_{s,t}$ be the broom consisting of $s+t$ edges obtained by identifying the center of $K_{1,s}$ with an end-vertex of P_{t+1}.

Theorem 2.23 ([127]) *For n sufficiently large,*

$$\frac{1}{2}nr_1 + c_k \leq ar(K_n, B_{s,t}) \leq \frac{1}{2}nr_2 + 1$$

where $r_1 = \max\{s-1, 2\lfloor (t-1)/2 \rfloor\}$, $r_2 = \max\{s-1, t\}$ and c_k does not depend on n.

Jiang [122]and Montellano-Ballesteros [186] independently found the anti-Ramsey number for stars, improving upon bounds in [175].

Theorem 2.24 ([122, 186])

$$ar(K_n, K_{1,k}) = \left\lfloor \frac{n(k-2)}{2} \right\rfloor + \left\lfloor \frac{n}{n-k+2} \right\rfloor$$

or possibly this value plus one if certain conditions hold.

In [175], Manoussakis, Spyratos, Tuza and Voigt found the number for spanning rainbow stars.

Theorem 2.25 ([175])

$$ar(K_n, K_{1,n-1}) = \frac{n(n-3)}{2} + \left\lfloor \frac{n}{3} \right\rfloor + 1.$$

Also in [186], the author found the anti-Ramsey numbers for stars $K_{1,k}$ in host graphs such as the hypercube Q_n, the grid $C_m \times C_n$ and a general graph G with $\delta(G) \geq k+4$. Similarly in [191], the authors consider rainbow stars within chosen subsets of vertices in colored multigraphs.

The anti-Ramsey numbers for paths were considered by Simonovits and Sós [210].

Theorem 2.26 ([210]) *There exists a constant c such that if $t \geq 5$ and $n > ct^2$, then for $\epsilon = 0, 1$, we have*

$$ar(K_n, P_{2t+3+\epsilon}) = tn - \binom{t+1}{2} + 1 + \epsilon.$$

Yuan [228] settles and old conjecture posed by Erdős, Simonovits and Sós [67] almost fifty years ago.

Theorem 2.27 ([228]) *Let P_k be a path on k vertices and $\ell = \lfloor (k-1)/2 \rfloor$. If $n \geq k \geq 5$, then*

$$\mathrm{AR}\,(n, P_k) = \max\left\{ \binom{k-2}{2} + 1, \binom{\ell-1}{2} + (\ell-1)(n-\ell+1) + \epsilon \right\}$$

where $\epsilon = 1$ if k is odd and $\epsilon = 2$ otherwise.

Simonovits and Sós also defined the H_0 spectra of colorings as follows. Given a particular graph $H \subseteq K_n$ and a coloring ϕ_r of K_n using r colors, let $c(H;\phi_r)$ denote the number of colors on H. For a given graph H_0, define the spectrum to be

$$S(H_0; n, \phi_r) = \{i : H \sim H_0, c(H; \phi_r) = i\}.$$

Let $\mathcal{T}_n$ be the set of all trees on n vertices and let $\mathcal{T}_n^*$ be the set of all graphs obtained from graphs in $\mathcal{T}_n$ be the removal of a single edge. Then the following was proven by Bialostocki and Voxman [32].

Theorem 2.28 ([32])

$$ar(K_n, \mathcal{T}_n) - ex(n, \mathcal{T}_n^*) = 1.$$

For a given set $S \subset \{1, \ldots, r\}$, a general question is whether or not there exists a coloring ϕ_r of K_n such that $S(H_0; n, \phi_r) = S$. Some cases of this problem are considered in [210] and it is noted that this is a generalization of work presented in [57]

2.4 Matchings

In 2004, Schiermeyer observed the following easy proposition. Unfortunately, it became a rather difficult problem to pin down the exact anti-Ramsey numbers for matchings.

Proposition 2 ([206])

$$ex(n, (k-1)K_2) + 1 \leq ar(K_n, kK_2) \leq ex(n, kK_2).$$

The extremal number for a matching is known from [62] to be as follows.

Theorem 2.29 ([62])

$$ex(n, kK_2) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}.$$

Also in [206], Schiermeyer used a counting technique to show that the lower bound is, in fact, the correct number for all $k \geq 2$ and $n \geq 3k+3$. This was later improved by Fujita, Kaneko, Schiermeyer and Suzuki [84] for all $n \geq 2k+1$. For $k = 2, 3, 4$ the same result was proven by Kaneko, Saito, Schiermeyer and Suzuki [139]. Finally, Chen, Li and Tu [52] used the Gallai-Edmonds Structure Theorem for matchings to prove the following, which shows that the lower bound of Proposition 2 is almost always the correct number.

Theorem 2.30 ([52])

$$ar(K_n, kK_2) = \begin{cases} 4, & n = 4 \textit{ and } k = 2, \\ ex(n, (k-1)K_2) + 2, & n = 2k \textit{ and } k \geq 7, \\ ex(n, (k-1)K_2) + 1, & \textit{otherwise.} \end{cases}$$

Haas and Young [110] verified a conjecture from [84] in the following result.

Theorem 2.31 ([110]) *For $k \geq 3$, if M_k is a matching on k edges,*

$$ar(K_{2k}, M_k) = \max\left\{\binom{2k-3}{2} + 3, \binom{k-2}{2} + k^2 - 2\right\}.$$

Others have studied rainbow matchings in bipartite graphs. Li, Tu and Jin [164] determined the anti-Ramsey number for matchings in complete bipartite graphs as follows.

Theorem 2.32 ([164]) *For all $m \geq n \geq k \geq 3$,*

$$ar(K_{m,n}, kK_2) = m(k-2) + 1.$$

In looking at more sparse graphs, Li and Xu [163] determined the anti-Ramsey number for matchings in m-regular bipartite graphs of order $2n$, denoted $B_{n,m}$.

Theorem 2.33 ([163]) *For all $k \geq 2$ and $m \geq 3$, if $n > (3k-1)$, then*

$$ar(B_{n,m}, kK_2) = m(k-2) + 1.$$

Gilboa and Roditty [93] proved reduction results of the form 'if $\text{ar}(K_n, L \cup t_1 P_s) \leq f(n, t_1, L)$ then $\text{ar}(K_n, L \cup tP_s) \leq f(n, t, L)$' where $s = 2$ or 3. These results lead to the following.

Corollary 2.34 ([93]) *For sufficiently large n,*

- $ar(K_n, P_3 \cup tP_2) = (t-1)(n - t/2) + 2$ *for* $t \geq 2$,
- $ar(K_n, P_4 \cup tP_2) = t(n - (t+1)/2) + 2$ *for* $t \geq 1$,
- $ar(K_n, C_3 \cup tP_2) = t(n - (t+1)/2) + 2$ *for* $t \geq 1$,
- $ar(K_n, tP_3) = (t-1)(n - t/2) + 2$ *for* $t \geq 1$,
- $ar(K_n, P_{k+1} \cup tP_3) = (t + \lfloor k/2 \rfloor - 1)(n - \frac{t + \lfloor k/2 \rfloor}{2}) + 2 + k \bmod 2$ *for* $k \geq 3$ *and* $t \geq 0$,
- $ar(K_n, P_2 \cup tP_3) = (t-1)(n - t/2) + 3$ *for* $t \geq 1$,
- $ar(K_n, kP_2 \cup tP_3) = (t + k - 2)(n - (t + k - 1)/2) + 2$ *for* $k \geq 2$ *and* $t \geq 2$.

2.5 Linear Forests

Fang, Győri, Lu, and Xiao [76] studied the anti-Ramsey number of star forests.

Theorem 2.35 ([76]) *Let $F = \bigcup_{i=1}^{t} K_{1,p_i}$ be a star forest, where $p_1 \geq 3, p_1 \geq p_2 \geq \cdots \geq p_t \geq 1$. Let $s = \max\{i : p_i \geq 2, 1 \leq i \leq t\}$. For $n \geq 3t^2(p_1+1)^2$, we have*

$$\begin{aligned}&\text{ar}(K_n, F)\\ &= \max\left\{\max_{1 \leq i \leq s}\left\{(i-1)n - \binom{i}{2} + \left\lfloor\frac{(p_i-2)(n-i+1)}{2}\right\rfloor + 1\right\}, (t-2)n - \binom{t-1}{2} + r\right\},\end{aligned}$$

where $r = 1$ if $p_{t-1} = 1$ and $r = 2$ otherwise.

If $p_1 = p_2 = \cdots = p_t = p \geq 3$ in Theorem 2.35, then the following corollary is immediate.

Corollary 2.36 ([76]) *Let $t \geq 2$, $p \geq 3$, for $n \geq 3t^2(p+1)^2$, we have*

$$\begin{aligned} \mathrm{ar}\left(K_n, tK_{1,p}\right) &= (t-1)n - \binom{t}{2} + \left\lfloor \frac{(p-2)(n-t+1)}{2} \right\rfloor + 1 \\ &= (t-1)n - \binom{t}{2} + \mathrm{ar}\left(K_{n-t+1}, K_{1,p}\right). \end{aligned}$$

They considered the anti-Ramsey number of linear forests and had the following two theorems.

Theorem 2.37 ([76]) *Let F be a linear forest with components of order $p_1, p_2, \ldots, p_t$, where $t \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq t$. Let $s = \sum_{i=1}^{t} \lfloor p_i/2 \rfloor - 2$.*

(1) *If all p_i are even, for n sufficiently large, we have*

$$sn - \binom{s-1}{2} + 1 \leq \mathrm{ar}\left(K_n, F\right) \leq sn - \binom{s+1}{2} + 2.$$

(2) *If all p_i are odd, for n sufficiently large, we have*

$$\mathrm{ar}\left(K_n, F\right) = (s+1)n - \binom{s+2}{2} + 1.$$

Theorem 2.38 ([76]) *Let F be a linear forest with components of order $p_1, p_2, \ldots, p_t$, where $t \geq 1, p_i \geq 2$ for $1 \leq i \leq t$ and at least one p_i is even. Then*

$$\mathrm{ar}\left(K_n, F\right) = \left(\sum_{i=1}^{t} \left\lfloor \frac{p_i}{2} \right\rfloor - 2\right).n + O(1)$$

Jiang and West [129] conjectured that:

Conjecture 3

$$\mathrm{ar}\left(K_n, T_k\right) \leq \frac{k-2}{2}n + O(1).$$

Notice that if T_k is a star or a path of even length, then $\mathrm{ar}\left(K_n, T_k\right) = \frac{k-2}{2}n + O(1)$.

A tree is called a *double star*, if there are exactly two vertices with degree at least 2. The two vertices are the centers of the double star. For $p \geq q \geq 1$, we denote a double star by $S_{p,q}$, if there are p leafs and q leafs incident to the two centers respectively.

Fang, Győri, Lu, and Xiao [76] computed the anti-Ramsey number of double stars for large n.

Theorem 2.39 *For $p \geq 2, 1 \leq q \leq p$ and $n \geq 6\left(p^2 + 2p\right)$, we have*

$$\mathrm{ar}\left(K_n, S_{p,q}\right) = \begin{cases} \left\lfloor \frac{(p-1)n}{2} \right\rfloor + 1, & if \ 1 \leq q \leq p-1 \\ \left\lfloor \frac{p(n-1)}{2} \right\rfloor + 1, & if \ q = p. \end{cases}$$

Notice that if we take $T_k = S_{p,p-1}$, where $k = 2p$, then we have $\mathrm{ar}\left(K_n, T_k\right) = \left\lfloor \frac{k-2}{2} \right\rfloor \frac{n}{2} + O(1)$.

2.6 Other Graphs

In full generality, Erdős, Simonovits and Sós [67] proved the following proposition which cements the relationship between the anti-Ramsey numbers and the extremal numbers of Turán. For this statement, given a set of graphs $\mathscr{H}$, let $\text{ex}(G, \mathscr{H})$ be the maximum number of edges in a subgraph of G containing no copy of H for any $H \in \mathscr{H}$.

Proposition 3 ([67]) *Given graphs G and H, we have*

$$ex(G, \mathscr{H}) + 1 \leq ar(G, H) \leq ex(G, H),$$

where $\mathscr{H} = \{H - e : e \in E(H)\}$.

For graphs containing at least one vertex of degree 2, Jiang [121] proved the following theorem.

Theorem 2.40 ([121]) *Given a graph H, let $\mathscr{H} = \{H - v : v \in V(H), d_H(v) = 2\}$ and suppose H has p vertices and q edges. For all positive integers n, we have*

$$ar(K_n, H) \leq ex(n, \mathscr{H}) + bn,$$

where $b = \max\{2p - 2, q - 2\}$.

This eventually led to the following result for subdivided graphs.

Theorem 2.41 ([121]) *If H is a graph containing at least two cycles in which each edge is incident to a vertex of degree two, then*

$$ar(K_n, H) = ex(n, \mathscr{H})(1 + o(1)),$$

where $\mathscr{H} = \{H - e : e \in E(H)\}$.

Theorem 2.40 also implies the following result from [17].

Theorem 2.42 ([17])

$$ar(K_n, K_{2,t}) = ex(K_n, K_{2,t-1}) + O(n).$$

A related result was shown in [148].

Theorem 2.43 ([148]) *For all $s \leq t$, there exists $c = c(s, t)$ such that*

$$ar(K_n, K_{s,t}) - ex(K_n, K_{s,t-1}) < cn.$$

In [17], the authors also provided the following general results for finding $K_{2,t}$ in complete bipartite graphs.

Theorem 2.44 ([17])

$$ar(K_{m,n}, K_{2,t}) = ex(K_{m,n}, K_{2,t-1}) + O(m + n).$$

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Theorem 2.45 ([17])

$$ar(K_{n,n}, K_{2,t}) = \sqrt{t-2}\,n^{3/2} + O(n^{4/3}).$$

Theorem 2.45 follows immediately from Theorem 2.44 and the following result of Füredi.

Theorem 2.46 ([90])

$$ex(K_{n,n}, K_{2,t}) = \sqrt{t-1}\,n^{3/2} + O(n^{4/3}).$$

For a C_4 with a single chord, which we will denote D for diamond, the following result was proven in [187, 188].

Theorem 2.47 ([187, 188]) *For $n \geq 4$,*

$$ex(K_n, \{C_3, C_4\}) + 1 \leq ar(K_n, D) \leq ex(K_n, \{C_3, C_4\}) + n.$$

For a cycle with a pendant edge, denoted by C_k^+, Gorgol showed the following interesting result.

Theorem 2.48 ([94]) *For $n \geq k+1$,*

$$ar(K_n, C_k^+) = ar(n, C_k).$$

If you add one additional pendant to the cycle, creating a graph denoted by C_k^{++}, Gorgol showed the following.

Theorem 2.49 ([94]) *For $n \geq k+2$,*

$$ar(K_n, C_k^{++}) > ar(n, C_k).$$

The proof of this result involves a slight modification of the coloring providing the lower bound of Conjecture 1.

A related result for a triangle with two pendant edges off a single vertex is the following.

Theorem 2.50 ([98]) *For $n \geq 5$,*

$$ar(K_n, K_{1,4} + e) = n + 1.$$

Let Q_n be the hypercube of dimension n, i.e. the graph of order 2^n in which the vertices are binary n-tuples and two vertices are adjacent if and only if the corresponding tuples differ by one term. Regarding the hypercube, Axenovich, Harborth, Kemnitz, Möller and Schiermeyer [14] provided a collection of results for finding one hypercube in another.

Theorem 2.51 ([14])

$$n2^{n-1} - \left\lfloor \frac{n}{k}(2^{n-1} - k + 1) \right\rfloor \leq ar(Q_n, Q_k) \leq n2^{n-1}\left(1 - \frac{n-k}{(n-1)k2^{k-2}}\right).$$

More specifically, the authors also proved the following.

Theorem 2.52 ([14])

$$ar(Q_n, Q_{n-1}) = \begin{cases} n2^{n-1} - 4 & \textit{for } n = 3, 4, 5, \\ n2^{n-1} - 3 & \textit{for } n \geq 6. \end{cases}$$

and $ar(Q_4, Q_2) = 18$.

Bode et al. [35] provide exact results for $\mathrm{ar}(Q_5, Q_2)$ and $\mathrm{ar}(Q_5, Q_3)$.

In other work, Gorgol and Łazuka computed the following anti-Ramsey numbers for stars with an added edge.

Theorem 2.53 ([97]) *For all* $n \geq 4$,

$$ar(K_n, K_{1,3} + e) = n - 1,$$

and for all $n \geq 5$,

$$ar(K_n, K_{1,4} + e) = n + 1.$$

We say that a graph H is doubly edge-critical if $\chi(H \setminus e) \geq p+1$ for any edge $e \in E(H)$ and there exists a pair of edges e, f for which $\chi(H \setminus \{e, f\}) = p$. Jiang and Pikhurko [125] obtained exact values of $ar(K_n, H)$ for doubly edge-critical graphs H and classified all sharpness examples. This result generalizes Theorem 2.18 since K_{p+2} is doubly edge-critical.

The *cyclomatic number* of a connected graph G, denoted $v(G)$, is the minimum number of edges that must be removed from G to make the resulting graph acyclic, that is, $v(G) = |E(G)| - |V(G)| + 1$.

Theorem 2.54 ([208]) *Let* H *be a connected graph of order* $p \geq 4$ *and cyclomatic number* $v(H) \geq 2$*. Then* $ar(K_n, H)$ *cannot be bounded from above by a function which is linear in* n.

Theorem 2.55 ([208]) *Let* H *be a graph of order* $p \geq 5$ *and cyclomatic number* $v(H) = 1$*. If* H *contains a cycle with* k *vertices for some* k *with* $3 \leq k \leq p - 2$*, then*

$$\left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2} + \left\lceil \frac{n}{k-1} \right\rceil - 1 \leq ar(K_n, H) \leq (p-2)n - p \cdot \frac{p-3}{2} - 1,$$

where $n \geq p$ *and* r *is the residue of* $n \mod k-1$.

Let B be the *bull* graph, the unique graph on 5 vertices with degree sequence $(1, 1, 2, 3, 3)$.

Theorem 2.56 ([208]) $ar(K_5, B) = 5$ *and* $ar(K_n, B) = n + 1$ *for* $n \geq 6$.

Gorgol and Görlich considered anti-Ramsey numbers for disjoint copies of a graph G, denoted by pG.

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Theorem 2.57 ([96]) *For any graph G on $n \geq 3$ vertices and for any p, we have*

$$ar(m, pG) \geq \max\left\{\binom{pn-2}{2} + 1, ar(m-p+1, G) + (p-1)m - \binom{p}{2}\right\}.$$

The same authors also offered the following conjecture.

Conjecture 4 ([96]) *For any graph G on $n \geq 3$ vertices and for any p, if $m \geq p|V(G)|$, then*

$$ar(m, pG) = ar(m-p+1, G) + (p-1)m - \binom{p}{2}$$

if and only if G is a tree.

For specific graphs, the following result was shown.

Theorem 2.58 ([96]) *For any integer $m \geq 6$, we have*

$$ar(m, 2P_3) = \begin{cases} 7 & \text{if } m = 6, \\ m & \text{if } m \geq 7. \end{cases}$$

For any integer $m > 12$, we have

$$ar(m, 3P_3) = 2m - 2.$$

For a graph G and a family $\mathcal{H}$ of graphs, the anti-Ramsey number $ar(G, \mathcal{H})$ is the maximum number k such that there exists an edge-coloring of G with exactly k colors without rainbow copy of any graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we denote $ar(G, \{H\})$ by $ar(G, H)$.

Fang, Győri, Lu, Xiao [77] studied the anti-Ramsey numbers of $\{C_3, C_4\}$ and C_3 in complete r-partite graphs and proved the following two theorems.

Theorem 2.59 ([77]) *For $r \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$, we have*

$$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, \{C_3, C_4\}) = n_1 + n_2 + \cdots + n_r - 1.$$

Theorem 2.60 ([77]) *For $r \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$, we have*

$$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, C_3) = \begin{cases} n_1n_2 + n_3n_4 + \cdots + n_{r-2}n_{r-1} + n_r + \frac{r-1}{2} - 1, & r \text{ is odd}; \\ n_1n_2 + n_3n_4 + \cdots + n_{r-1}n_r + \frac{r}{2} - 1, & r \text{ is even}. \end{cases}$$

They generalize the theorem of Alon [3] to complete r-partite graphs. We call two subgraphs H_1 and H_2 of G *independent* if they are vertex disjoint.

Theorem 2.61 ([77]) *For $r \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_r \geq 1$, we have*

$$\mathrm{ar}(K_{n_1,n_2,\ldots,n_r}, C_4) = n_1 + n_2 + \cdots + n_r + t - 1,$$

where $t = \min\left\{\left\lfloor \frac{\sum_{i=1}^r n_i}{3} \right\rfloor, \left\lfloor \frac{\sum_{i=2}^r n_i}{2} \right\rfloor, \sum_{i=3}^r n_i\right\}$ is the maximum number of independent triangles of $K_{n_1,n_2,\ldots,n_r}$.

Notice that $K_n + \overline{K_s}$ is a complete $(n+1)$-partite graph $K_{s,1,\ldots,1}$. If $2s \geq n \geq 4$, then the maximum number of independent triangles of $K_n + \overline{K_s}$ is $\lfloor \frac{n}{2} \rfloor$. They had the following corollary and this answers a question of Gorgol [95].

Corollary 2.62 ([77]) *For $2s \geq n \geq 4$, we have* $\mathrm{ar}(K_n + \overline{K_s}, C_4) = \lfloor \frac{3n}{2} \rfloor + s - 1$.

Jin, Zhong, and Sun [133] studied the anti-Ramsey numbers of C_3 and C_3^+ in a complete multipartite graph.

Theorem 2.63 ([133]) *Let $m \geq 3$ and $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$. Then*

$$ar(K_{n_1,n_2,\ldots,n_m}, C_3) = \begin{cases} \sum_{i=1}^{k} n_{2i-1}n_{2i} + k - 1, & \text{if } m = 2k; \\ \sum_{i=1}^{k} n_{2i-1}n_{2i} + n_{2k+1} + k - 1, & \text{if } m = 2k+1. \end{cases}$$

Theorem 2.64 ([133]) *Let $m \geq 3, n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$ and $n = n_1 + n_2 + \cdots + n_m \geq 4$. Then*

$$ar(K_{n_1,n_2,\ldots,n_m}, C_3^+) = ar(K_{n_1,n_2,\ldots,n_m}, C_3).$$

2.7 Planar Anti-Ramsey Numbers

The study of planar anti-Ramsey numbers $ar_{\mathcal{P}}(n, H)$ was initiated by Horňák, Jendrol, Schiermeyer and Soták [118] (under the name of rainbow numbers).

For two positive integers a and b, we use $a\ mod(b)$ to denote the remainder when a is divided by b.

Horňák, Jendrol, Schiermeyer and Soták derived the following results for cycles.

Theorem 2.65 ([118]) *Let n, k be positive integers.*

(a) $ar_{\mathcal{P}}(n, C_3) = \lfloor (3n-6)/2 \rfloor$ *for* $n \geq 4$.

(b) $ar_{\mathcal{P}}(n, C_4) \leq 2(n-2)$ *for* $n \geq 4$, *and* $ar_{\mathcal{P}}(n, C_4) \geq (9(n-2) - 4r)/5$ *for* $n \geq 42$ *and* $r = (n-2)mod(20)$.

(c) $ar_{\mathcal{P}}(n, C_5) \leq 5(n-2)/2$ *for* $n \geq 5$, *and* $ar_{\mathcal{P}}(n, C_5) \leq (19(n-2) - 10r)/9$ *for* $n \geq 2$ *and* $r = (n-2)mod(18)$.

(d) $ar_{\mathcal{P}}(n, C_k) \geq (3n-6) \cdot \frac{k-3}{k-2} - \frac{2k-7}{k-2}$ *for* $6 \leq k \leq n$.

Lan, Shi, and Song proved [150] new lower bounds for $ar_{\mathcal{P}}(n, C_k)$ when $k \geq 5$.

Theorem 2.66 ([150]) *For integers $k \geq 5$, $n \geq k^2 - k$, and $r = (n-2)mod(k^2-k-2)$,*

$$ar_{\mathcal{P}}(n, C_k) \geq \left(\frac{k-3}{k-2} + \frac{2}{3(k+1)(k-2)}\right)(3n-6) - \frac{2k^2-5k-5}{k^2-k-2}r.$$

Theorem 2.67 ([150]) *Let $n \geq 119$ be an integer and let $r = (n+7)mod(18)$. Then* $ar_{\mathcal{P}}(n, C_5) \geq (39n - 123 - 21r)/18$.

Theorem 2.68 ([150]) *For integers $k \geq 5$ and $q \geq k-1$,* $\lfloor \frac{2k-7}{k-3} q \rfloor \leq ar_{\mathcal{P}}(W_q, C_k) \leq \lfloor \frac{2k-5}{k-2} q \rfloor$.

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Corollary 2.69 ([150]) *Let $k \geq 5$ and $q \geq k-1$ be integers. If $q \in \{t(k-2), \ldots, t(k-2)+k-4-t\}$ for some integer $t \in [k-4]$, then $ar_{\mathcal{P}}(W_q, C_k) = 2q - \lfloor \frac{q}{k-3} \rfloor$.*

Theorem 2.70 ([150]) *For integers $q \geq 5$, $ar_{\mathcal{P}}(W_q, C_6) = \lfloor 5q/3 \rfloor$.*

They obtained new upper bounds for $ar_{\mathcal{P}}(n, C_6)$ when $n \geq 8$ and $ar_{\mathcal{P}}(n, C_7)$ when $n \geq 13$, respectively.

Theorem 2.71 ([150]) *$ar_{\mathcal{P}}(n, C_6) \leq 17(n-2)/6$ for all $n \geq 8$ and $ar_{\mathcal{P}}(n, C_7) \leq (59n - 113)/20$ for all $n \geq 13$.*

Theorem 2.72 ([150]) *For any $k \in \{8, 9\}$, let $\varepsilon = k mod(2)$ and $n \geq k$ be an integer. Then $ar_{\mathcal{P}}(n, P_k) \geq (3n + 3\varepsilon - \varepsilon^* - 3)/2$, where $\varepsilon^* = (n + 1 + \varepsilon) mod(2)$.*

A lower bound for $ar_{\mathcal{P}}(n, P_k)$ when $k \geq 10$ is also given.

Theorem 2.73 ([150]) *Let k and n be two integers such that $n \geq k \geq 10$. Let $\varepsilon = k mod(2)$. Then*

$$ar_{\mathcal{P}}(n, P_k) \geq \begin{cases} n + 2k - 12, & if\ \ k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5, \\ (3n + 9\lfloor k/2 \rfloor + 3\varepsilon - 43)/2, & if\ \ 3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15, \\ 2n + k - 14, & if\ \ n > 5\lfloor k/2 \rfloor + \varepsilon - 15. \end{cases}$$

2.8 Anti-Ramsey Numbers in Matroids

Given a matroid $M = (E, \mathcal{I})$, where E is the (colored) ground set and $\mathcal{I}$ is the family of independent sets. A *rainbow basis* is a maximum independent set in which each element receives a different color. The *rank* of a set $S \subseteq E$, written by $r_M(S)$, is the maximum size of an independent set in S. A *flat* F is a maximal set in M with a fixed rank. The *anti-Ramsey* number of t pairwise disjoint rainbow bases in M, denoted by $\mathrm{ar}(M, t)$, is defined as the maximum number of colors m such that there exists an m coloring of the ground set E of M which contains no t pairwise disjoint rainbow bases.

Lu and Meier [170] obtained the following results.

Theorem 2.74 ([170]) *For any matroid $M = (E, \mathcal{I})$ with rank at least 2, if there is a flat F of M satisfying $|E| - |F| < t\,(r_M(E) - r_M(F))$, then $\mathrm{ar}(M, t) = |E|$. Otherwise,*

$$\mathrm{ar}(M, t) = \max_{F : r_M(F) \leq r_M(E) - 2} \{|F| + t\,(r_M(E) - r_M(F) - 1)\},$$

where the maximum is taken among all flats F of M with $r_M(F) \leq r_M(E) - 2$.

They also included the elementary result in the case where the rank of M is 1. Given a matroid M define M_0 as all rank 0 elements.

Theorem 2.75 ([170]) *For any matroid $M = (E, \mathcal{I})$ of rank 1, we have*

$$\mathrm{ar}(M, t) = \begin{cases} |E| & \textit{if } |E| < |M_0| + t, \\ |M_0| + t - 1 & \textit{otherwise.} \end{cases}$$

When $r_M(E) = 2, r_M(F) = 0$ implies $F \subseteq M_0$, while $r_M(F) = 1$ implies $F \subseteq cl(x)$, the closure of some rank 1 element x.

Corollary 2.76 ([170]) *For any matroid $M = (E, \mathcal{I})$ of rank 2, we have*

$$\operatorname{ar}(M,t) = \begin{cases} |E| & \text{if } |E| < |M_0| + 2t \text{ or } |E| < |cl(x)| + t \text{ for some } x, \\ |M_0| + t & \text{otherwise.} \end{cases}$$

2.9 Anti-Ramsey Numbers of Hypergraphs

A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ is a finite set $V(\mathcal{H})$ of elements, called vertices, together with a finite set $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$, called *hyperedges* or simply edges. The union of hypergraphs $\mathcal{G}$ and $\mathcal{H}$ is the graph $\mathcal{G} \cup \mathcal{H}$ with vertex set $V(\mathcal{G}) \cup V(\mathcal{H})$ and edge set $E(\mathcal{G}) \cup E(H)$. If each edge of $\mathcal{H}$ has exactly r vertices, $\mathcal{H}$ is called r*-uniform*. For a subset V' of $V(\mathcal{H})$, denoted by $\mathcal{H}[V']$ the subhypergraph of $\mathcal{H}$ induced by V'. For $v \in V(\mathcal{H})$, we use $\mathcal{H} - v$ to denote $\mathcal{H}[V(\mathcal{H}) \setminus \{v\}]$. For an edge e in $E(\mathcal{H})$, denoted by $\mathcal{H} - e$ the hypergraph obtained by deleting e from $\mathcal{H}$. For a vertex $v \in V(\mathcal{H})$, the degree $d_{\mathcal{H}}(v)$ is defined as the number of edges of $\mathcal{H}$ containing v. A vertex of degree zero is called an *isolated vertex*. For $u, v \in V(\mathcal{H})$, we define $d_{\mathcal{H}}(u, v)$ to be the number of edges of $\mathcal{H}$ containing $\{u, v\}$, and we call this number the co-degree of $\{u, v\}$. For a hypergraph $\mathcal{H}$, we denote the number of edges in $\mathcal{H}$ by $e(\mathcal{H})$. A complete r-uniform hypergraph is a hypergraph whose edge set consists of all r-subsets of the vertex set. A matching in a hypergraph is a set of edges in which no two edges have a common vertex. We call a matching with k edges a k*-matching*, denoted by M_k. An edge-colored hypergraph is called *rainbow hypergraph* if the all of its edges have different colors. The representing hypergraph of a hypergraph $\mathcal{H}$ with an edge coloring c is a spanning subhypergraph of $\mathcal{H}$ obtained by taking one edge of each color of c. For an edge set $E \subseteq E(\mathcal{H})$, let $c(E)$ denote the set of colors of edges in E. For simplicity, when $E = \{e\}$ and $E = E(\mathcal{H})$, we use $c(e)$ and $c(\mathcal{H})$ instead of $c(\{e\})$ and $c(E(\mathcal{H}))$, respectively.

Let $n_1, n_2, ..., n_r$ be integers and $V_1, V_2, ..., V_r$ be disjoint vertex sets with $|V_i| = n_i$ for each $i = 1, 2, ..., r$. A complete r-partite r-uniform hypergraph on vertex classes $V_1, V_2, ..., V_r$, denoted by $K_{n_1,...,n_r}$, is defined to be the r-uniform hypergraph whose edge set consists of all the r-element subsets S of $V_1 \cup \ldots \cup V_r$ such that $|S \cap V_i| = 1$ for all $i = 1, 2, ..., r$.

Given two hypergraphs $\mathcal{H}$ and $\mathcal{G}$, the *anti-Ramsey number* of $\mathcal{H}$ in $\mathcal{G}$, denoted by $ar(\mathcal{G}, \mathcal{H})$, is the maximum number of colors in a coloring of the edges of $\mathcal{G}$ with no rainbow copy of $\mathcal{H}$. When $\mathcal{G}$ is an r-uniform complete hypergraph on n vertices, $ar_r(\mathcal{G}, \mathcal{H})$ is the anti-Ramsey number of $\mathcal{H}$. The *Turán number* $ex_r(\mathcal{G}, \mathcal{H})$ is the maximum number of edges in an $\mathcal{H}$-free subhypergraph of $\mathcal{G}$, where $\mathcal{H}$-free hypergraph is one which contains no $\mathcal{H}$ as a subhypergraph.

Let $\mathcal{H}$ be an s-uniform hypergraph.

(i) A *Berge path of length* k in $\mathcal{H}$ is a family of k distinct edges $e_1, \ldots, e_k$ and $k+1$ distinct vertices $v_1, \ldots, v_{k+1}$ such that for each $1 \le i \le k, e_i$ contains v_i and v_{i+1}. Let $\mathcal{B}_k$ denote the family of Berge paths of length k. A Berge cycle of length k in $\mathcal{H}$ is a cyclic list of k distinct edges $e_1, \ldots, e_k$ and k distinct vertices $v_1, \ldots, v_k$ such that e_i contains v_i and v_{i+1} for each $1 \le i \le k$, where $v_{k+1} = v_1$. Denote the family of all Berge cycles of length k by $\mathcal{BC}_k$.

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(ii) A *loose path of length* k in $\mathcal{H}$ is a collection of distinct edges $\{e_1, e_2, \ldots, e_k\}$ such that consecutive edges intersect in at least one element and nonconsecutive edges are disjoint. Denote the family of loose paths of length k by $\mathcal{P}_k$. A loose cycle is defined similarly in a cyclic order, and denote the family of all loose cycles of length k by $\mathcal{C}_k$.

(iii) A *linear path of length* k in $\mathcal{H}$ is a collection of distinct edges $\{e_1, e_2, \ldots, e_k\}$ such that consecutive edges intersect in exactly one element and nonconsecutive edges are disjoint. Let $\mathbb{P}_k$ denote the linear path of length k. A linear cycle is defined similarly in a cyclic order, and let $\mathbb{C}_k$ denote the collection of linear path of length k.

Gu, Li, and Shi [101] gave the exact anti-Ramsey numbers of short paths $\mathbb{P}_i, \mathcal{B}_i, \mathcal{P}_i$ for $i = 2, 3$.

Theorem 2.77 ([101]) (i) *For* $s \geq 3$ *and* $n \geq 3s-4, \mathrm{ar}(n, s, \mathbb{P}_2) = 2$.
(ii) *For* $s \geq 4$ *and sufficiently large* n, *ar* $(n, s, \mathbb{P}_3) = \binom{n-2}{s-2} + 2$.
(iii) *For* $n \geq 3s-4, \mathrm{ar}(n, s, \mathcal{B}_2) = \mathrm{ar}(n, s, \mathcal{P}_2) = 2$.
(iv) *For* $n \geq 4s-3, \mathrm{ar}(n, s, \mathcal{B}_3) = \mathrm{ar}(n, s, \mathcal{P}_3) = 3$.

For linear paths and loose paths, they [101] obtained the exact anti-Ramsey numbers for sufficiently large n.

Theorem 2.78 ([101]) *For any integer* k, *if* $k = 2t \geq 4$ *and* $s \geq 3$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t+1 > 5$ *and* $s \geq 4$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

Theorem 2.79 ([101]) *For any integer* k, *if* $k = 2t \geq 4$ *and* $s \geq 3$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t+1 \geq 5$ *and* $s \geq 3$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3.$$

Theorem 2.80 ([101]) *For any integer* k, *if* $k = 2t \geq 8$ *and* $s \geq 4$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathbb{C}_k) = \mathrm{ar}(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t+1 \geq 11$ *and* $s \geq k+3$, *then for sufficiently large* n,

$$\mathrm{ar}(n, s, \mathbb{C}_k) = \mathrm{ar}(n, s, \mathbb{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + \binom{n-t-1}{s-2} + 2.$$

Theorem 2.81 ([101]) *For any integer k, if $k = 2t \geq 8$ and $s \geq 4$, then for sufficiently large n,*

$$\mathrm{ar}(n, s, \mathcal{C}_k) = \mathrm{ar}(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 2;$$

if $k = 2t + 1 \geq 11$ and $s \geq k + 3$, then for sufficiently large n,

$$\mathrm{ar}(n, s, \mathcal{C}_k) = \mathrm{ar}(n, s, \mathcal{P}_k) = \binom{n}{s} - \binom{n-t+1}{s} + 3.$$

Theorem 2.82 ([101]) *If $k > 2s + 1$, then for sufficiently large n,*

$$\frac{2n}{k}\binom{\lfloor k/2 \rfloor}{s} \leq \mathrm{ar}(n, s, \mathcal{B}_k) \leq \frac{n}{k-1}\binom{k-1}{s} + 1$$

If $s + 2 \leq k \leq 2s + 1$, then for sufficiently large n,

$$\frac{n}{s+1}\left\lfloor \frac{k-2}{2} \right\rfloor \leq \mathrm{ar}(n, s, \mathcal{B}_k) \leq \frac{n}{k-1}\binom{k-1}{s} + 1.$$

If $k \leq s + 1$, then for sufficiently large n,

$$\frac{n}{s+1}\left\lfloor \frac{k-2}{2} \right\rfloor \leq \mathrm{ar}(n, s, \mathcal{B}_k) \leq \frac{(k-2)n}{s+1} + 1.$$

Proposition 4 ([101]) *For any fixed integers $s \geq 4, k \geq 3$,*

$$\mathrm{ex}(n, s, \mathcal{B}_{k-1}) + 2 \leq \mathrm{ar}(n, s, \mathcal{BC}_k) \leq \mathrm{ex}(n, s, \mathcal{B}_{k-1}) + k.$$

Xue, Shan, and Kang [227] provided a lower and upper bound.

Proposition 5 ([227]) $ex_r(K_{n_1,\ldots,n_r}, M_{k-1}) + 1 \leq ar_r(K_{n_1,\ldots,n_r}, M_k) \leq ex_r(K_{n_1,\ldots,n_r}, M_k)$.

Theorem 2.83 ([227]) *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq k \geq 1$ and $r \geq 2$, every subhypergraph of $K_{n_1,\ldots,n_r}$ with $(k-1)n_2 \cdots n_r$ edges and without isolated vertices, except for $K_{k-1,n_2,\ldots,n_r}$, contains a k-matching.*

Xue, Shan, and Kang [227] focused on the anti-Ramsey number of k-matchings in complete r-partite r-uniform hypergraphs.

Theorem 2.84 ([227]) (i) *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 3$,*

$$ar_r(K_{n_1,\ldots,n_r}, M_2) = 1.$$

(ii) *For $n_1 = 2$, let t be the largest integer such that $n_t = n_1 = 2$. Then*

$$ar_r(K_{n_1,\ldots,n_r}, M_2) = 2^{t-1}.$$

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Theorem 2.85 ([227]) *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 2k-1$ and $k \geq 3$,*

$$ar_r(K_{n_1,\dots,n_r}, M_k) = (k-2)n_2 \cdots n_r + 1.$$

Moreover, every $((k-2)n_2 \cdots n_r + 1)$-edge-coloring except for ϕ_r of $K_{n_1,\dots,n_r}$ contains a rainbow k-matching.

Corollary 2.86 ([227]) *For $n_r \geq n_{r-1} \geq \cdots \geq n_1 \geq 2k-1$ and $k \geq 2$,*

$$ar_r(K_{n_1,\dots,n_r}, M_k) = ex_r(K_{n_1,\dots,n_r}, M_{k-1}).$$

Let $ar_3(m_1, m_2, m_3, k)$ denote anti-Ramsey number of a k-matching in the complete tripartite 3-uniform hypergraph $\mathcal{H}$.

Jin [130] got the following results.

Theorem 2.87 ([130]) *For $m_1, m_2, m_3 \geq 2$,*

$$ar_3(m_1, m_2, m_3, 2) = \begin{cases} 4, & if m_1 = m_2 = m_3 = 2; \\ 2, & if m_1 = m_2 = 2, m_3 \geq 3; \\ 1, & if m_1 \geq 2, m_2, m_3 \geq 3. \end{cases}$$

Theorem 2.88 ([130]) *For $3k-2 \leq m_1 \leq m_2 \leq m_3$ and $k \geq 3$,*

$$ar_3(m_1, m_2, m_3, k) = m_2 m_3 (k-2) + 1.$$

2.10 Generalizations of Anti-Ramsey Theory

For given integers p and q, a (p,q)-coloring of K_n is a coloring in which the edges of every K_p subgraph uses at least q colors. Let $f(n,p,q)$ be the minimum number of colors in a (p,q)-coloring of K_n. When $q = 2$, this reduces down to the classical anti-Ramsey problem

This problem was first considered by Elekes, Erdős and Füredi in [59] and later revisited by Erdős and Gyárfás in [63]. They used the Local Lemma to provide the following general upper bound on $f(n,p,q)$.

Theorem 2.89 ([63]) *For some c depending only on p and q,*

$$f(n,p,q) \leq cn^{(p-2)/\left[\binom{p}{2} - q + 1\right]}.$$

In general, they also proved the following.

Theorem 2.90 ([63]) *For any p, let $q = \binom{p}{2} - p + 3$. Then $f(n,p,q)$ is linear in n while $f(n,p,q-1)$ is sublinear (less or equal to $cn^{1-1/(p-1)}$).*

The authors also considered $f(n,p,q)$ for many small fixed values of p and q. In particular, they struggled with finding upper and lower bounds on $f(n,5,9)$ and $f(n,4,3)$. Since then, Mubayi [194], Axenovich [10] and Krop and Krop [147] have improved the bounds on these small cases.

In similar work, Axenovich, Füredi and Mubayi [13] studied the function $r(G,H,q)$ which is defined to be the minimum number of colors in a coloring of G in which every copy of $H \subseteq G$ together receive at least q colors. The paper includes a variety of results concerning the case when G and H are complete bipartite graphs. Li, Broersma and Wang [157] studied the function $r(K_{n,n}, K_{s,t}, q)$. Mubayi and West also considered bipartite graphs in [195]. Improvements were made by Ling in [165].

As another variation, Axenovich and Kündgen [21] defined the function $R(n,p,q_1,q_2)$ to be the maximum number of colors in a coloring of K_n such that the number of colors used on a subgraph A is between q_1 and q_2 (inclusive) for every subgraph A with $|A| = p$. The case when $q_1 = 1$ reduces to the problem studied in [67]. In [21], the authors mention general results from [1] for a variety of values for p, q_1, q_2 and bounds are proven for $p = 4$, $q_2 = 4$ or 5. A similar problem, when $q_1 = 2$, $q_2 = |E(H)| - 1$ and we restrict only the copies of H in a general graph G, was considered in [15] under the title of mixed anti-Ramsey numbers.

The problem of finding the minimum number of colors $f(n,e,L)$ necessary to color a graph on n vertices and e edges such that *every* copy of L has all edges of different colors was studied in [7, 44, 204]. In [44], the following question was asked.

Question 1 *Let L be a connected bipartite graph that is not a star. Is it true that*

$$\lim_{n\to\infty} \frac{f(n,\alpha n^2, L)}{n} = \infty?$$

This question is answered in the affirmative in [204] for the case when L is a connected, bipartite graph that is not complete bipartite. The function $f(n,e,L)$ was also studied in [45] where some bounds were provided for some classes of graphs L. These are roughly described as follows:

- A lower bound when L is bipartite with $\Delta(L) \geq 2$ and having at least two strongly independent edges (meaning that the end vertices of the edges induce no other edges),

- A lower bound when L has two strongly independent edges and is not a disjoint union of cliques, or

- An upper bound when L has no two strongly independent edges.

A similar question was studied for sub-hypercubes of a hypercube in [7].

Let $f(n)$ denote the largest number of edges in a rainbow subgraph of a properly edge-colored complete graph on n vertices. Then the following was shown in [24].

Theorem 2.91 ([24])

$$(2n)^{1/3} \leq f(n) \leq 8(n \ln n)^{1/3}.$$

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Within the class of graphs in which each vertex is incident to many colors, the anti-Ramsey problem was studied in [19] under the title of local anti-Ramsey numbers. Many general bounds are presented in [19] while another specific number can be found in [87].

Haxell and Kohayakawa [115] considered an anti-Ramsey type problem for finding rainbow cycles in colored graphs with large girth.

Define the *size anti-Ramsey number* of a graph H, denoted $AR_s(H)$, to be the smallest number of edges in a graph G such that any proper edge-coloring of G contains a rainbow copy of H. The size anti-Ramsey number was originally defined in [20] along with several notions of online anti-Ramsey numbers. Relationships between these numbers as well as general bounds were also proven in [20]. The general behavior of $AR_s(K_k)$ was settled in [4] with the following result, answering a question from [20].

Theorem 2.92 ([4])

$$AR_s(K_k) = \Theta(k^6/\log^2 k).$$

Also introduced in [4] is the concept of *degree anti-Ramsey number* of a graph H, denoted $AR_d(H)$, to be the minimum value of d such that there is a graph G with maximum degree at most d such that any proper edge-coloring of G contains a rainbow copy of H. Some observations about the function $AR_d(H)$ are also included in [4].

In [200] and [135], the problem of vertex-coloring plane graphs avoiding rainbow faces is discussed. Other vertex colorings related to anti-Ramsey theory were discussed in, for example, [149, 193] and many others. In [185] the author discusses anti-Ramsey concepts for finding rainbow colored edge-cuts.

The anti-Ramsey problem has also been studied in a variety of other contexts. For hypergraphs, see [8, 153, 198]. For random graphs, see [37, 143, 144, 202]. For anti-Ramsey in groups, see [25, 209]. For directed graphs, see [28] among others. Concerning integers and rainbow arithmetic progressions, see [12, 136, 137, 138].

3 Rainbow Ramsey Theory

3.1 Classical Rainbow Ramsey Numbers

Definition 2 *For given two graphs G_1, G_2, the rainbow Ramsey number (also sometimes called the constrained Ramsey number)* $\mathrm{RR}(G_1, G_2)$ *is defined to be the minimum integer N such that any edge-coloring of the complete graph K_N using any number of colors must contain either a monochromatic copy of G_1 or a rainbow copy of G_2.*

Although commonly called constrained Ramsey numbers, we use the term rainbow Ramsey numbers to describe this concept in following the notation of [50]. In [119], the following is proven.

Theorem 3.1 ([119]) *The rainbow Ramsey number* $\mathrm{RR}(G_1, G_2)$ *exists if and only if G_1 is a star or G_2 is a forest.*

As observed in [119], $\mathrm{RR}(G_1, K_{1,t+1})$ is equivalent to the t-local Ramsey number of G_1, introduced in [103, 105]. In [31], the following is proven.

Theorem 3.2 ([31]) *For every positive integer n,*

$$\mathrm{RR}(nK_2, nK_2) = n(n-1) + 2.$$

More generally, for $G_1 = nK_2$, $G_2 = mK_2$, the following natural conjecture is proposed in [73].

Conjecture 5 ([73]) *For any two integers n, m with $n \geq 3, m \geq 2$,*

$$\mathrm{RR}(nK_2, mK_2) = m(n-1) + 2.$$

We can easily see that $m(n-1)+2 \leq \mathrm{RR}(nK_2, mK_2) \leq 2(n-1)m$, when $n \geq 2$. For the lower bound, consider a coloring of the graph $K_{m(n-1)+1}$ as follows. Color all of the edges of a subgraph isomorphic to K_{2n-1} with color 1. Choose $n-1$ additional vertices and color all of the edges among these vertices and between these vertices and those already colored with color 2. For each color $i = 3, 4, \ldots, m-1$, choose $n-1$ additional vertices and color the edges among those vertices and between those vertices and the part of the graph already colored with color i. The resulting graph has $2n-1+(m-2)(n-1) = m(n-1)+1$ vertices and contains no set of n independent edges in the same color. Since only $m-1$ colors appear, it also can not contain a set of m independent edges in different colors.

For the upper bound, notice that it holds for $n = 2$ and for $m = 1$ provided $n \geq 2$. For any $n \geq 3$ and $m \geq 2$, suppose $\mathrm{RR}(nK_2, (m-1)K_2) \leq 2(n-1)(m-1)$ and $\mathrm{RR}((n-1)K_2, mK_2) \leq 2(n-2)m$. Consider any edge-coloring of $K_{2(n-1)m}$. If the resulting graph does not contain a rainbow mK_2, then without loss of generality it must contain a monochromatic $(n-1)K_2$. If we remove these $2(n-1)$ vertices, there are $2(n-1)(m-1)$ vertices remaining. Thus, there is either a monochromatic nK_2 or a rainbow $(m-1)K_2$ on the remaining vertices. Without loss of generality, we have a monochromatic $(n-1)K_2$, say in color c, and a disjoint rainbow $(m-1)K_2$. Either the rainbow $(m-1)K_2$ contains an edge in color c or it does not. If it contains an edge in color c, then this edge along with the monochromatic $(n-1)K_2$ form a monochromatic nK_2. Otherwise, an edge in color c from the $(n-1)K_2$ may be added to the rainbow $(m-1)K_2$ to produce a rainbow mK_2.

In attempts to prove this conjecture the following results are proven in [73].

Theorem 3.3 ([73]) *For any two positive integers n, m with $2 \leq m < n$,*

$$\mathrm{RR}(nK_2, mK_2) = m(n-1) + 2.$$

Theorem 3.4 ([73]) $\mathrm{RR}(3K_2, 4K_2) = 10$ *and* $\mathrm{RR}(4K_2, 5K_2) = 17$.

Theorem 3.5 ([73]) *For $n > 5$ and $2 \leq m \leq \frac{3}{2}(n-1)$,*

$$\mathrm{RR}(nK_2, mK_2) = m(n-1) + 2.$$

Next we deal with the case where G_1 is a star and G_2 is a matching. In [74], the following results are proven. First there is a general lower bound.

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Theorem 3.6 ([74]) *For any positive integers n and m, provided that n is odd or m is even,*

$$\mathrm{RR}(K_{1,n}, mK_2) \geq (n-1)(m-1) + 2.$$

If n is even and m is odd, then

$$\mathrm{RR}(K_{1,n}, mK_2) \geq (n-1)(m-1) + 1.$$

Next, the authors found the following upper bounds.

Theorem 3.7 ([74]) *For any positive integers n and m,*

$$\mathrm{RR}(K_{1,n}, mK_2) \leq (n-1)(m-1) + 2 + \binom{m-n+3}{2}.$$

Theorem 3.8 ([74]) *For any positive integers n and m,*

$$\mathrm{RR}(K_{1,n}, mK_2) \leq (n+1)(m-1) + 2.$$

If $(n+1)(m-1) \geq 2m+1$ (for instance, $n \geq 2$ and $m \geq 4$ or $n \geq 3$ and $m \geq 3$), then we may improve the bound above to $\mathrm{RR}(K_{1,n}, mK_2) \leq (n+1)(m-1)$.

More specifically, the following special case is proven.

Theorem 3.9 ([74]) $\mathrm{RR}(K_{1,3}, 3K_2) = 7$.

On the other hand, when $G_1 = nK_2, G_2 = K_{1,m}$, the following upper and lower bounds are proven in [74].

Theorem 3.10 ([74]) *For any positive integers $n \geq 2$ and $m \geq 3$,*

$$\mathrm{RR}(nK_2, K_{1,m}) \geq (2m-3)(n-1) + 1.$$

Theorem 3.11 ([74]) *For any integers m and n, where $m \geq 2$ and $n \geq 2$,*

$$\mathrm{RR}(nK_2, K_{1,m}) \leq m(m-1)n - \frac{1}{2}(3m+1)(m-2).$$

Finally, we conclude this section by considering the case where G_2 is a path. Let $\mathrm{R}(G,G), \mathrm{R}(G,G,G)$ be the 2- and 3-coloring Ramsey numbers of G respectively. In [104], the authors proved the following general results which relate rainbow Ramsey numbers to classical graph Ramsey numbers.

Theorem 3.12 ([104]) *For every graph G of order $n \geq 5$,*

$$\mathrm{RR}(G, P_4) = \mathrm{R}(G,G).$$

Theorem 3.13 ([104]) *For $n \geq 3$,*

$$\mathrm{RR}(P_n, P_5) = \mathrm{R}(P_n, P_n, P_n).$$

Theorem 3.14 ([104]) *If G is a connected non-bipartite graph then*

$$\mathrm{RR}(G, P_5) = \mathrm{R}(G, G, G).$$

Theorem 3.15 ([104]) *For $n \geq 3$,*

$$\mathrm{RR}(C_n, P_5) = \mathrm{R}(C_n, C_n, C_n).$$

Let T_5 be the tree obtained from $K_{1,3}$ by subdividing one edge.

Theorem 3.16 ([104]) *If $G = P_n$ or $C_n (n \geq 3)$ or G is non-bipartite and connected, then*

$$\mathrm{RR}(G, T_5) = \mathrm{RR}(G, P_5).$$

Alon, Jiang, Miller and Pritikin [6] provided general bounds on the rainbow Ramsey number for a star versus a complete graph.

Theorem 3.17 ([6]) *For some absolute constants c_1 and c_2, for $t \geq 3$,*

$$\frac{c_1 m t^3}{\ln t} \leq \mathrm{RR}(K_{1,m+1}, K_t) \leq \frac{c_2 m t^3}{\ln t}.$$

Some similar work has been done with rainbow Ramsey in algebraic structures in [79, 184, 168, 183] among many others.

3.2 Bipartite Rainbow Ramsey Numbers

Given two bipartite graphs G_1 and G_2, the bipartite rainbow Ramsey number $\mathrm{BRR}(G_1, G_2)$ is the smallest integer N such that any coloring of the edges of $K_{N,N}$ with any number of colors contains a monochromatic copy of G_1 or a rainbow copy of G_2. For the existence of $\mathrm{BRR}(G_1, G_2)$, the following is proved in [75]:

Theorem 3.18 ([75]) *The bipartite rainbow Ramsey number $\mathrm{BRR}(G_1, G_2)$ exists if and only if G_1 is a star or G_2 is a star forest (i.e., a union of stars).*

Let $\mathcal{S}_r$ denote any star forest with r components and let S_r, B_r, T_r, and $\mathcal{F}_r$ be any star forest, bipartite graphs, tree or forest, respectively with r edges. In [75], the following general bounds for the bipartite rainbow Ramsey number are established. We first consider the case when the rainbow graphs of interest are general bipartite graphs.

Theorem 3.19 ([75]) *Let G_n be any connected bipartite graph for which the largest partite set has n vertices. If $BRR(G_n, B_m)$ exists, then*

$$\mathrm{BRR}(G_n, B_m) \geq (n-1)(m-1) + 1.$$

This result easily implies the following two corollaries.

Corollary 3.20 ([75])

$$\mathrm{BRR}(K_{1,n}, mK_2) \geq (n-1)(m-1)+1.$$

Corollary 3.21 ([75])

$$\mathrm{BRR}(T_n, S_m) \geq \left(\left\lceil \frac{n+1}{2} \right\rceil - 1 \right) (m-1)+1.$$

We next consider the case when the rainbow graph is a forest.

Theorem 3.22 ([75]) *If G_n is any forest with n nontrivial components, then*

$$\mathrm{BRR}(G_n, S_m) \geq (n-1)(m-1)+1.$$

This result immediately implies the next two corollaries.

Corollary 3.23 ([75])

$$\mathrm{BRR}(nK_2, K_{1,m}) \geq (n-1)(m-1)+1.$$

Corollary 3.24 ([75])

$$\mathrm{BRR}(nK_2, S_m) \geq (n-1)(m-1)+1.$$

The next two corollaries take the previous bounds and provide exact results.

Corollary 3.25 ([75])

$$\mathrm{BRR}(K_{1,n}, K_{1,m}) = (n-1)(m-1)+1.$$

Corollary 3.26 ([75])

$$\mathrm{BRR}(nK_2, mK_2) = (n-1)(m-1)+1.$$

Theorem 3.27 ([75]) *Suppose $\mathcal{F}_m$ has no isolated vertices. Then*

$$\mathrm{BRR}(K_{1,n}, \mathcal{F}_m) = O(mn),$$
$$\mathrm{BRR}(T_n, K_{1,m}) = O(mn).$$

Corollary 3.28 ([75])

$$\mathrm{BRR}(\mathcal{F}_n, K_{1,m}) = O(mn).$$

Regarding matchings and stars, the following general results were proven in [75].

Theorem 3.29 ([75]) *For any integers $n, m \geq 2$,*

$$(n-1)(m-1)+1 \leq \mathrm{BRR}(K_{1,n}, mK_2) \leq n(m-1)+1.$$

Furthermore, if $n > 2$, the upper bound can be improved to $n(m-1)$.

Theorem 3.30 ([75]) *For positive integers $n \geq 3$ and $1 \leq m \leq n+2$,*

$$\mathrm{BRR}(K_{1,n}, mK_2) = (n-1)(m-1)+1.$$

Theorem 3.31 ([75]) *For any integers $n, m \geq 2$,*

$$\mathrm{BRR}(nK_2, K_{1,m}) \geq \max(2(n-1)(m-2)+1, (n-1)(m-1)+1).$$

Theorem 3.32 ([75]) *For any integers $n \geq 2$ and $m \geq 3$,*

$$\mathrm{BRR}(nK_2, K_{1,m}) \leq (3m-5)(n-1)+1.$$

More specifically, for a small star or a small matching, the following cases were shown.

Theorem 3.33 ([75]) *For any integer $n \geq 2$,*

$$\mathrm{BRR}(nK_2, K_{1,2}) = n.$$

Theorem 3.34 ([75])

$$\mathrm{BRR}(2K_2, K_{1,m}) = 2m-3.$$

Concerning stars and paths, the following results were shown.

Theorem 3.35 ([75]) *For integers $n, m \geq 3$,*

$$2(\lceil n/2 \rceil - 1)(m-2) \leq \mathrm{BRR}(P_{n+1}, K_{1,m}) \leq (n-1)(m-1).$$

Theorem 3.36 ([75]) *For any integers $n, m \geq 2$,*

$$\begin{array}{rcl}(n-1)(m-1)+1 & \leq & \mathrm{BRR}(K_{1,n}, P_{m+1}) \\ & \leq & \max((m-2)(n-1) + \lceil \frac{m+2}{2} \rceil, (m-1)(n-1)+1).\end{array}$$

This bound implies the following exact result for small paths.

Corollary 3.37 ([75]) *For integers $n, m \geq 2$ such that $m \leq 2n-3$,*

$$\mathrm{BRR}(K_{1,n}, P_{m+1}) = (n-1)(m-1)+1.$$

Theorem 3.38 ([75]) *For any integer $p \geq 1$,*

$$\mathrm{BRR}(K_{1,2}, P_{4p-1}) \geq 4p+1.$$

Also, in [26], the following is shown.

Theorem 3.39 ([26])

$$\mathrm{BRR}(K_{1,n}, K_{2,2}) = 3n-2.$$

Theorem 3.40 ([218]) *For fixed $t \geq 3$, $s \geq (t-1)!+1$ and n large,*

$$\mathrm{BRR}(K_{s,t}, K_{1,n}) = \Theta(n^t).$$

Theorem 3.41 ([218]) *For $n > t \geq 3$,*

$$t^2(n-1)+1 \leq \mathrm{BRR}(K_{1,n}, K_{t,t}) \leq t^3(n-1)+t-1.$$

Theorem 3.42 ([218]) *For $m = 2, 3, 5$, if $n \to \infty$, then*

$$\mathrm{BRR}(C_{2m}, K_{1,n}) \geq (1-o(n))n^{m/(m-1)}.$$

Theorem 3.43 ([218]) *For any integers $n, m \geq 2$,*

$$(2m-1)(n-1)+1 \leq \mathrm{BRR}(K_{1,n}, C_{2m}) \leq 2m(n-2)+\frac{1}{2}m(m-1)(n-1)+2.$$

Let $B_{s,t}$ denote the broom with s edges in the star part and t edges in the path part.

Theorem 3.44 ([218]) *For any integers $n, s, t \geq 2$,*

$$\max\{(n-1)(s+\lceil t/2\rceil-1), 2(n-2)(\lceil t/2\rceil-1)\}-1 \leq \mathrm{BRR}(B_{s,t}, K_{1,n}) \leq (2s+t-3)(n-1).$$

Theorem 3.45 ([218]) *For any integers $n, s, t \geq 2$,*

$$(n-1)(s+t-1)+1 \leq \mathrm{BRR}(K_{1,n}, B_{s,t}) \leq (n-1)(s+t-1)+s+\frac{t+1}{2}.$$

3.3 Pattern Ramsey Theory

A color pattern is defined to be a graph with colored edges. A family of patterns $\mathscr{F}$ is called a Ramsey family if there exists an integer n_0 such that in every coloring of the edges of K_n with $n \geq n_0$, there exists some pattern in $\mathscr{F}$.

Definition 3 *The* pattern Ramsey number *for a Ramsey family $\mathscr{F}$ of patterns is the smallest integer n_0 such that in every coloring of K_n with $n \geq n_0$, there exists some pattern in $\mathscr{F}$.*

Notice that this definition is closely related to the definition of the rainbow Ramsey number except, as opposed to restricting attention to monochromatic or rainbow graphs, one is allowed to choose any coloring. Also note another similarity in that the number of colors is unlimited.

In 1950, Erdős and Rado [64] classified which families of patterns are Ramsey. A coloring of a graph is said to be lexical if there exists an ordering of the vertices (left to right) such that two edges get the same color if and only if they share a right endpoint. For ease of notation, let H^{mono}, H^{rain} and H^{lex} be the monochromatic, rainbow and lexical colorings of H respectively. Erdős and Rado showed the following.

Theorem 3.46 ([64, 65]) *There is a constant C_p such that every coloring of $E(K_n)$ for $n > C_p$ contains a K_p that is monochromatic, rainbow or lexically colored.*

The result was actually proven for hypergraphs. This result was just the beginning of a very difficult problem, the problem of finding such pattern Ramsey numbers. In honor of Erdős and Rado, the pattern Ramsey number for finding a monochromatic, rainbow or lexical complete graph of order k is commonly denoted $ER(k)$. The original proof in [64] provides an upper bound and Galvin [71] (p. 30) noticed a lower bound on $ER(k)$. These bounds were improved by Lefmann and Rödl in [151] but in [152], Lefmann and Rödl provided a new proof of Theorem 3.46 and better bounds on $ER(k)$ (again for hypergraphs).

Theorem 3.47 ([152])

$$2^{c_1 k^2} \leq ER(k) \leq 2^{c_2 k^2 \log k}$$

for some constants c_1 and c_2.

Similarly, in [151], Lefmann and Rödl considered finding ordered rainbow paths or monochromatic complete graphs in edge-colored totally ordered complete graphs. They proved that the necessary order is related to the classical Ramsey numbers for finding a path or a complete graph.

Concerning the more general problem of considering general patterns of colors, Jamison and West [120] considered a particular family of colorings (equipartitioned stars) while Axenovich and Jamison [16] studied another family ($\mathscr{F} = \{K_n^{lex}, K_3^{rain}, H^{mono}\}$). Notice this is related to the rainbow triangle free work discussed in Section 4. The following 3 results were proven in this work. Define $f(n, H)$ to be the smallest integer m such that every coloring of K_m contains either a K_n^{lex}, a K_3^{rain} or H^{mono}.

Theorem 3.48 ([16])

$$f(n, H) \leq 3^{n|H|}.$$

Theorem 3.49 ([16]) *For any connected graph H and any n, there is a constant $c = c(n)$ such that $f(n, H) \leq cR_{n-1}(H)$ (the classical $n-1$ color Ramsey number for H).*

Theorem 3.50 ([16])

$$5^{\lfloor n/2 \rfloor - 1} + 1 \leq f(n, K_3) \leq 5^{n/2}.$$

Similar coloring problems were considered in [153, 201] for finding colored subsets in a k-uniform hypergraph. Another bound on some such Ramsey numbers can be found in [123]. The authors of [29] also found Mixed Pattern Ramsey numbers for a rainbow or monochromatic triangle after exclusion of colored graphs H where H is any colored 4-cycle, almost any colored 4-clique and bounds when H is a monochromatic odd cycle or a star when the number of colors is sufficiently large.

4 Gallai-Ramsey Theory

4.1 Gallai Colorings

The avoidance of rainbow colored subgraphs began with Gallai in [91] where the author studied transitively orientable graphs. The results contained in [91] were reproduced in [108] where Gyárfás and Simonyi translated them to the terminology of graph coloring.

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Definition 4 *A coloring of a complete graph G is said to be a* Gallai coloring *if this coloring contains no rainbow triangle.*

Gyárfás and Simonyi restated the following theorem attributed to Gallai and also to Cameron and Edmonds in [46].

Theorem 4.1 ([46, 91, 108]) *Any Gallai coloring can be obtained by substituting Gallai colored complete graphs into the vertices of a 2-colored complete graph.*

This theorem follows from the lemma below, which provides another useful description of Gallai colorings.

Lemma 1 ([91, 108]) *Every Gallai coloring with at least three colors has a color which spans a disconnected graph.*

One may also note that Theorem 4.1 is equivalent to the following result.

Theorem 4.2 *In any Gallai colored complete graph, there exists a partition of the vertices (called a* Gallai partition*) such that there are at most two colors on the edges between the parts and only one color on edges between each pair of parts.*

Using Theorem 4.1, Cameron, Edmonds and Lovász [47] proved the following extension of the Perfect Graph Theorem (see [169]).

Theorem 4.3 ([47]) *Let G be a Gallai 3-coloring of a complete graph. If the graphs induced on two of the colors are perfect, then the graph induced on the third color is also perfect.*

In the same note, the authors go on to conjecture that if K_n is 3-colored so that no configuration from a given class occurs and two of the colors induce perfect graphs, then the third color also induces a perfect graph. This conjecture follows from the Strong Perfect Graph Theorem which was proven in [56] and is stated as follows.

Theorem 4.4 (Strong Perfect Graph Theorem [56]) *A graph is perfect if and only if no induced subgraph is an odd cycle of length at least 5 or the complement of one.*

In honor of Berge who conjectured the above, the class of graphs having no such induced odd cycle or its complement have been called *Berge graphs*.

Also using Theorem 4.1, Gyárfás and Simonyi prove the following three results, each of which extends older results from 2-colorings to Gallai colorings. For the first result, a broom is a path with a star at one end. Although the following three results are also stated in [140], we present them here for the sake of completeness.

Theorem 4.5 ([108]) *In every Gallai coloring of a complete graph, there exists a spanning monochromatic broom.*

This result generalizes a result of Burr [43] which states that any 2-colored complete graph contains a monochromatic spanning broom. Theorem 4.6 generalizes a result of Bialostocki, Dierker and Voxman [30] who proved the same result in the case of 2-colored complete graphs.

Theorem 4.6 ([108]) *In every Gallai coloring, there is a monochromatic spanning tree with height at most two.*

Theorem 4.6 is proven, as in the following, using the structure provided by Theorem 4.1.

Theorem 4.7 ([108]) *Any Gallai coloring of the complete graph K_n contains a monochromatic star S_t for some $t \geq \frac{2n}{5}$.*

Theorem 4.7 is sharp by the following construction. Consider 5 copies of $K_{n/5}$ labeled as $G_0, G_1, \dots, G_4$ each colored entirely in color 1. Color the edges between G_i and G_{i+1} with color 2 and color all edges between G_i and G_{i+2} with color 3 for all i (modulo 5). This graph contains the aforementioned monochromatic star but not a larger one.

Theorems 4.5, 4.6 and 4.7 answer questions posed by Bialostocki and Voxman in [33].

More recently, Gyárfás, Sárközy, Sebő and Selkow [106] provided even more monochromatic structure in Gallai colorings. A double star is defined to be a tree of diameter 3, or in other words, two disjoint stars with centers joined by an edge. In the first result, they extend Theorem 4.7 to a double star.

Theorem 4.8 ([106]) *Every Gallai coloring of K_n contains a monochromatic double star with at least $\frac{3n+1}{4}$ vertices. This is asymptotically best possible.*

Continuing in the tradition of extending 2-coloring results to Gallai colorings, the authors also extend results from [61] and [102] which find a monochromatic diameter 2 subgraph of a 2-colored complete graph.

Theorem 4.9 ([106]) *In every Gallai coloring G of K_n, there is a monochromatic diameter 2 subgraph with at least $\left\lceil \frac{3n}{4} \right\rceil$ vertices. This is best possible for every n.*

This result is best possible by the following construction. Consider a 2-coloring of K_4 in which each color is isomorphic to P_4. Then substitute an equal (or as close to equal as possible) number of vertices for each vertex of the K_4. The coloring of these new blocks is arbitrary. This construction contains a monochromatic diameter 2 subgraph of order $\left\lceil \frac{3n}{4} \right\rceil$ but no larger.

Others have studied *exact Gallai cliques* which are colorings of cliques in which every copy of a smaller clique has exactly a predetermined number of colors. In [57], an upper bound of approximately $5^{k/2}$ was found for the number of vertices in an exact Gallai clique using k colors. As an extension of Theorem 4.1, a characterization of exact Gallai cliques was given by Ball, Pultr and Vojtěchovský [27].

More specifically, Chung and Graham [57] studied the function $f(s,t;k)$ defined to be the largest value of m such that it is possible to k-color the edges of K_m so that every $K_s \subseteq K_m$ has exactly t different colors. Using this notation, the aforementioned result can be restated as:

Theorem 4.10 ([15, 57, 106])

$$f(3,2;k) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

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In [15] and [106], the previous result was stated in terms of Gallai colorings and proven using the decomposition from Theorem 4.1. Axenovich and Iverson, in [15], even classified all rainbow colorings of a complete graph with no rainbow or monochromatic triangle. In [57], Chung and Graham also proved the following.

Theorem 4.11 ([57]) *For $k \geq 4$,*

$$f(4, 3; k) = k + 2.$$

Furthermore, Chung and Graham also stated, without proof, the following two results. The first extends the previous theorem for all values of $s \leq k$ while the second demonstrates the extreme change in behavior if s is allowed to be larger than k.

Theorem 4.12 ([57]) *For $5 \leq s \leq k$,*

$$f(s, s-1; k) = k + 1.$$

Theorem 4.13 ([57])

$$(1 + o(1))k^2 \leq f(k+1, k; k) \leq k^2 + k.$$

The idea of Gallai colorings has also been considered in the context of multigraphs. In another generalization of Theorem 4.1, the authors of [114] provided a construction of all finite Gallai multigraphs similar to that of Gallai for graphs. Concerning Gallai multigraphs, Diwan and Mubayi asked the following question.

Question 2 ([58]) *Let R, G, and B be graphs on the same vertex set of size n. How large must $\min\{e(R), e(G), e(B)\}$ be to guarantee that $R \cup G \cup B$ contains a rainbow triangle?*

Using the partition result from [114], Magnant [172] recently provided the following solution to the question of Diwan and Mubayi in the case where the graph is large and complete, meaning that between every pair of vertices, there is at least one edge.

Let G be a G-colored multigraph on n vertices using three colors and let $m(G)$ be the minimum number of edges in a single color in G. Let M be limit of the maximum value of $m(G)$ over all G-colored, complete, multigraphs G on n vertices as $n \to \infty$. With this notation, the main result is the following.

Theorem 4.14 ([172])

$$M = \frac{26 - 2\sqrt{7}}{81} n^2 \sim 0.25566 n^2.$$

4.2 General Gallai-Ramsey Theory for Rainbow Triangles

In [106], the authors introduced a restricted Ramsey number which they called $RG(r,H)$ to be the minimum m such that in every Gallai coloring of K_m with r colors, there is a monochromatic copy of H. This concept naturally extends to any rainbow colored graph in the following sense.

Definition 5 *Given two graphs G and H, the k-colored* Gallai Ramsey number $\mathrm{gr}_k(G:H)$ *is defined to be the minimum integer n such that every k-coloring (using all k colors) of the complete graph on n vertices contains either a rainbow copy of G or a monochromatic copy of H.*

Notice that this definition is similar to the rainbow (or constrained) Ramsey numbers (see Definition 2) except, in this case, the number of colors is fixed. This definition is also very closely related to the function $MaxR(G,H)$ studied in [11, 15]. Essentially these functions are duals. Similarly, we define the following notation for when the number of colors used in the coloring is at most a fixed value k.

Definition 6 *Given two graphs G and H, the k-colored* upper Gallai Ramsey number $\mathrm{gr}'_k(G:H)$ *is defined to be the minimum integer n such that every k-coloring (using at most k colors) of the complete graph on n vertices contains either a rainbow copy of G or a monochromatic copy of H.*

Note that if $\mathrm{gr}_k(H:G)$ is a monotone increasing function of k (on an interval $a \leq k \leq b$), then these two functions will be equal (on the same interval). Somewhat surprisingly, this is not always the case (see Theorem 4.113 and Conjecture 12).

In particular, using this definition, Theorem 4.7 can be restated as follows, which was also noted in [182].

Theorem 4.15 ([182])

$$\mathrm{gr}'_k(K_3:S_t) = \begin{cases} \frac{5t-3}{2} & \text{for odd } t, \\ \frac{5t-6}{2} & \text{otherwise.} \end{cases}$$

Theorem 4.10 can also be restated in this notation. In [106], the authors provide the asymptotic behavior of $\mathrm{gr}_k(K_3:H)$ for a general graph H.

Theorem 4.16 ([106]) *Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $\mathrm{gr}_k(K_3:H)$ is exponential in k. If H is bipartite, then $\mathrm{gr}_k(K_3:H)$ is linear in k.*

The lower bound for the case when H is not bipartite comes from the following inductive construction. Certainly there exists a small graph in one color containing no H. Suppose there exists G_k using k colors which contains no monochromatic copy of H. Then let G_{k+1} be two copies of G_k with all possible edges in between using the new color. The graph G_{k+1} also contains no monochromatic copy of H. For the lower bound when H is bipartite, the construction involves adding vertices to the graph with all edges in a single color. If H is a

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star, this result becomes more complicated in light of the difference between $\mathrm{gr}_k(\cdot : \cdot)$ and $\mathrm{gr}'_k(\cdot : \cdot)$. Moreover, if H is not a star, Li and Wang [161] showed that $\mathrm{gr}_k(K_3 : H) = \mathrm{gr}'_k(K_3 : H)$.

In [78], Faudree, Gould, Jacobson and Magnant proved the following specific Gallai Ramsey numbers

Theorem 4.17 ([78])

(1) $\mathrm{gr}_k(K_3 : C_4) = k + 4$ *for* $k \geq 2$.

(2) $\mathrm{gr}_k(K_3 : P_4) = k + 3$ *for* $k \geq 1$.

(3) $\mathrm{gr}_k(K_3 : P_5) = k + 4$ *for* $k \geq 1$.

(4) $\mathrm{gr}_k(K_3 : P_6) = 2k + 4$ *for* $k \geq 1$.

The authors of [78] also found the Gallai Ramsey numbers for all trees of order at most 6. Regarding paths in general, the following represents the best known bounds.

Theorem 4.18 ([78], [113]) *Given integers* $n \geq 3$ *and* $k \geq 1$,

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq \mathrm{gr}_k(K_3 : P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3 \left\lceil \frac{n}{2} \right\rceil .$$

Regarding cycles, the following are the best known general bounds.

Theorem 4.19 ([86], [113]) (1) *Given integers* $n \geq 2$ *and* $k \geq 1$,

$$(n-1)k + n + 1 \leq \mathrm{gr}_k(K_3 : C_{2n}) \leq (n-1)k + 3n.$$

(2) *Given integers* $n \geq 2$ *and* $k \geq 1$,

$$n2^k + 1 \leq \mathrm{gr}_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

For specific small cycles, Fujita and Magnant obtained the following.

Theorem 4.20 ([86]) (1) *For any positive integer* k,

$$\mathrm{gr}_k(K_3 : C_6) = 2k + 4.$$

(2) *For any positive integer* $k \geq 2$,

$$\mathrm{gr}_k(K_3 : C_5) = 2^{k+1} + 1.$$

Bosse and Song [38] got the Gallai-Ramsey numbers of 7-cycle. Gregory, C. Magnant, and Z. Magnant [99] got the exact values for 8-cycle. For 10-cycle and 12-cycle, Lei, Shi, Song, Zhang [154] obtained their exact values. For 9-cycle and 11-cycle, Bosse and Song got their exact values. C. Bosse, Z. Song, and J. Zhang also obtained the Gallai-Ramsey numbers of C_{13} and C_{15}.

Chen, Li, and Pei [53] derived the following bounds.

Theorem 4.21 ([53]) *For any integers $k, n \geq 2$,*

$$n2^k + 1 \leq \mathrm{gr}_k(K_3 : C_{2k+1}) \leq 2^k(\log_2 n + 4)n.$$

Wang et al. [220] and Zhang et al. [230] confirmed that belief with the following.

Theorem 4.22 ([220, 230]) *For integers $\ell \geq 3$ and $k \geq 1$, we have*

$$\mathrm{gr}_k(K_3 : C_{2\ell+1}) = \ell \cdot 2^k + 1.$$

Fox, Grinshpun, and Pach [80] proposed the following conjecture.

Conjecture 6 ([80]) *For integers $k \geq 1$ and $t \geq 3$,*

$$\mathrm{gr}_k(K_3 : K_t) = \mathrm{gr}'_k(K_3 : K_t) = \begin{cases} (\mathrm{R}_2(K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even,} \\ (t-1) \cdot (\mathrm{R}_2(K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Conjecture 6 was verified for the cases $t = 3$ [15, 57, 106] and $t = 4$ [166]. Li, Broersma and Wang [158] obtained the following best until now upper bound for $t \geq 5$.

Theorem 4.23 ([158]) *For integers $k \geq 3$ and $t \geq 5$, we have*

$$\mathrm{gr}_k(K_3 : K_t) = \mathrm{gr}'_k(K_3 : K_t) < 2^{2k(t-2)-3}.$$

Recall that the *Ramsey number* $r(p, q)$ is the minimum integer n such that, for every coloring of the edges of the complete graph on n vertices, using red and blue, there is either a red clique of order p, or a blue clique of order q. In particular, we write $r(p) = r(p, p)$.

Conjecture 7 ([80]) *For $k \geq 1$ and $p \geq 3$,*

$$\mathrm{gr}_k(K_3 : K_p) = \begin{cases} (r(p) - 1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (p-1)(r(p) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The case where $p = 3$ was actually verified in 1983 by Chung and Graham [54]. A simplified proof was given by Gyárfás et al. [106].

Theorem 4.24 ([54, 106]) *For $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Liu, Magnant, Saito, Schiermeyer, and Shi proved the following result, the first open case of Conjecture 7.

Theorem 4.25 ([166]) *For $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : K_4) = \begin{cases} 17^{k/2} + 1 & \text{if } k \text{ is even,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

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Define the *refined k-colored Gallai-Ramsey number* $gr_k(K_3 : H_1, H_2, \ldots, H_k)$ to be the minimum number of vertices n such that, every k-coloring of the complete graph on n vertices contains either a rainbow triangle, or a copy of H_i in color i, for some i. Since we will generally be working only with K_4 and K_3, for an integer s with $0 \le s \le k$, we use the following shorthand notation

$$\mathrm{gr}_k(K_3 : sK_4, (k-s)K_3) = \mathrm{gr}_k(K_3 : K_4, K_4, \ldots, K_4, K_3, K_3, \ldots, K_3)$$

where we look for K_4 in any of the first s colors or K_3 in any of the remaining $k-s$ colors.

In order to prove Theorem 4.25, they actually proved the following refined version. Theorem 4.25 follows as a corollary to Theorem 4.26 by choosing $s = k$.

Theorem 4.26 ([166]) *Let $k \ge 1$, and s be an integer with $0 \le s \le k$. Then*

$$\mathrm{gr}_k(K_3 : sK_4, (k-s)K_3) = g(k, s)$$

where

$$g(k,s) = \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } s = k \text{ and } s \text{ is odd,} \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2} + 1 & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even.} \end{cases}$$

Magnant and Schiermeyer [181] proved Conjecture 7 in the case where $p = 5$. This result is particularly interesting since $R(K_5, K_5)$ is still not known. Let $R = R(K_5, K_5) - 1$ and note that the known bounds on this Ramsey number give us $42 \le R \le 47$.

Theorem 4.27 ([181]) *For any integer $k \ge 2$,*

$$\mathrm{gr}_k(K_3 : K_5) = \begin{cases} R^{k/2} + 1 & \text{if } k \text{ is even,} \\ 4 \cdot R^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

unless $R = 42$, in which case we have

$$\begin{cases} \mathrm{gr}_k(K_3 : K_5) = 43 & \text{if } k = 2, \\ 42^{k/2} + 1 \le gr_k(K_3 : K_5) \le 43^{k/2} + 1 & \text{if } k \ge 4 \text{ is even,} \\ 169 \cdot 42^{(k-3)/2} + 1 \le gr_k(K_3 : K_5) \le 4 \cdot 43^{(k-1)/2} + 1 & \text{if } k \ge 3 \text{ is odd.} \end{cases}$$

Since we will generally be working only with K_5 or K_4 or K_3, for three integers r, s, t we use the following shorthand notation:

$$gr_k(K_3 : rK_5, sK_4, tK_t) = gr_k(K_3 : K_5, K_5, \ldots, K_5, K_4, K_4, \ldots, K_4, K_3, K_3, \ldots, K_3),$$

where we look for K_5 in any of the first r colors or K_4 in any of the s middle colors or K_3 in any of the last t colors.

To simplify the notation, we let c_1 denote the case where r, s, t are all even, c_2 denote the case where r, s are both even and t is odd, and so on for $c_3, \ldots, c_{11}$. For nonnegative integers r, s, t, let $k = r + s + t$. Then we define

$$gr_k(K_3 : rK_5, sK_4, tK_3) = \begin{cases} R^{r/2} \cdot 17^{s/2} \cdot 5^{t/2} + 1 \\ \quad \text{if } r, s, t \text{ are even, } (c_1) \\ 2 \cdot R^{r/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1 \\ \quad \text{if } r, s \text{ are even, and } t \text{ is odd, } (c_2) \\ 3 \cdot R^{r/2} \cdot 17^{(s-1)/2} + 1 \\ \quad \text{if } r \text{ is even, } s \text{ is odd, and } t = 0, (c_3) \\ 4 \cdot R^{(r-1)/2} + 1 \\ \quad \text{if } r \text{ is odd, and } s = t = 0, (c_4) \\ 8 \cdot R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1 \\ \quad \text{if } r \text{ is even, and } s, t \text{ are odd, } (c_5) \\ 13 \cdot R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1 \\ \quad \text{if } r, t \text{ are odd, and } s \text{ is even, } (c_6) \\ 16 \cdot R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-2)/2} + 1 \\ \quad \text{if } r, t \text{ are even, } t \geq 2, \text{ and } s \text{ is odd, } (c_7) \\ 24 \cdot R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{t/2} + 1 \\ \quad \text{if } r, s \text{ are odd, and } t \text{ is even, } (c_8) \\ 26 \cdot R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-2)/2} + 1 \\ \quad \text{if } r \text{ is odd, } s \text{ is even, } t \geq 2 \text{ is even, } (c_9) \\ 48 \cdot R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1 \\ \quad \text{if } r, s, t \text{ are odd, } (c_{10}) \\ 72 \cdot R^{(r-1)/2} \cdot 17^{(s-2)/2} + 1 \\ \quad \text{if } r \text{ is odd, } t = 0, \text{ and } s \geq 2 \text{ is even. } (c_{11}) \end{cases}$$

For ease of notation, let $g(r, s, t)$ be the value of $gr_k(K_3 : rK_5, sK_4, tK_3)$ claimed above. Also, for each i with $1 \leq i \leq 11$, let $g_i(r, s, t) = g(r, s, t) - 1$ in the case where (c_i) holds. Now we can state Theorem **??**.

Theorem 4.28 ([181]) *For nonnegative integers r, s, t, let $k = r + s + t$. Then*

$$gr_k(K_3 : rK_5, sK_4, tK_3) = g(r, s, t).$$

For the monochromatic star, Li and Wang [161] obtained the exact values of $\text{gr}_k(K_3 : K_{1,t})$ for any $t \geq 3$ and $k \geq 3$.

Theorem 4.29 ([161]) *For $t \geq 4$,*

$$\text{gr}_k(K_3 : K_{1,t}) = \begin{cases} (5t-3)/2, & \text{if } t \text{ is odd and } 3 \leq k \leq (5t-11)/2, \\ (5t-6)/2, & \text{if } t \text{ is even and } 3 \leq k \leq (5t-14)/2. \end{cases}$$

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Theorem 4.30 ([161]) *The following statements hold:*

(1) $\mathrm{gr}_3(K_3 : K_{1,3}) = 5$; $\mathrm{gr}_4(K_3 : K_{1,4}) = \mathrm{gr}_5(K_3 : K_{1,4}) = 7$; $\mathrm{gr}_9(K_3 : K_{1,6}) = 11$;

(2) *For* $n \geq 4$, *if* (i) $t = 3$ *and* $k \geq 4$, (ii) $t \geq 5$ *is odd and* $k \geq (5t - 9)/2$, (iii) $t = 4$ *and* $k \geq 6$, (iv) $t = 6$ *and* $k \geq 10$, *or* (v) $t \geq 8$ *is even and* $k \geq (5t - 12)/2$, *then there is either a rainbow* K_3 *or a monochromatic* $K_{1,t}$ *in any colored* K_n *using exactly* k *colors.*

For non-bipartite graphs, the picture is not clear. Given a graph H, call a graph H' a *reduction* of H if H' can be obtained from H by identifying sets of non-adjacent vertices (and removing any resulting repeated edges). Let $\mathscr{H}$ be the set of all possible reductions of H. For the sake of the following main definition, let $\mathrm{R}_2(\mathscr{H})$ be the minimum integer n such that every 2-coloring of K_n contains a monochromatic copy of some graph in the set $\mathscr{H}$. Since this quantity is bounded above by the Ramsey number $\mathrm{R}(H, H)$, its existence is obvious. Now a critical definition.

Definition 7 ([173]) *If* $\mathscr{H}$ *is the set of all reductions of a given graph* H, *define the function* $m(H)$ *to be*

$$m(H) = \mathrm{R}_2(\mathscr{H}).$$

Using this definition, a lower bound on the Gallai-Ramsey number for any non-bipartite graph H has been shown.

Theorem 4.31 ([173]) *For a connected non-bipartite graph* H *and an integer* $k \geq 2$, *we have that* $\mathrm{gr}_k(K_3 : H)$ *is at least*

$$\begin{cases} (\mathrm{R}(H,H) - 1) \cdot (m(H) - 1)^{(k-2)/2} + 1 & \text{if } k \text{ is even,} \\ (\chi(H) - 1) \cdot (\mathrm{R}(H,H) - 1) \cdot (m(H) - 1)^{(k-3)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

In summary of the known sharp values of Gallai-Ramsey numbers concerning rainbow triangles, we present the following tables. For the tables, we describe some special trees as follows:

- P_4^+ is the graph consisting of a P_4 with the addition of a pendant vertex adjacent to an interior vertex of the P_4,
- P_5^+ is the graph consisting of a P_5 with the addition of a pendant vertex adjacent to an interior vertex (but not the center) of the P_5,
- P_4^{++} is the graph consisting of a P_4 with the addition of two pendant vertices adjacent to different interior vertices of the P_4,
- P_4^{+2} is the graph consisting of a P_4 with the addition of two pendant vertices adjacent to a single interior vertex of the P_4,
- $P_5^{+'}$ is the graph consisting of a P_5 with the addition of a pendant vertex adjacent to the center vertex of the P_5,

- B_m is the book on m pages, or rather $K_2 + \overline{K_m}$.

Graph	$\mathrm{gr}_k(K_3 : H)$	Cite	Graph	$\mathrm{gr}_k(K_3 : H)$	Cite
P_4	$k+3$	[78]	C_4	$k+4$	[78]
P_5	$k+4$	[78]	C_5	$2^{k+1}+1$	[86]
P_6	$2k+4$	[78]	C_6	$2k+4$	[86]
P_7	$2k+5$	[174]	C_7	$3 \cdot 2^k + 1$	[42]
P_8	$3k+5$	[174]	C_8	$3k+5$	[99]
P_4^+	$k+4$	[78]	C_9	$4 \cdot 2^k + 1$	[39]
P_5^+	$k+5$	[78]	C_{11}	$5 \cdot 2^k + 1$	[39]
P_4^{++}	$2k+4$	[78]	C_{13}	$6 \cdot 2^k + 1$	[40]
P_4^{+2}	$2k+4$	[78]	C_{15}	$7 \cdot 2^k + 1$	[40]
$P_5^{+'}$	$k+5$	[78]			

Graph	$\mathrm{gr}_k(K_3 : H)$	Cite
K_3	$\begin{cases} 5^{k/2} & k \text{ even} \\ 2 \cdot 5^{(k-1)/2} & k \text{ odd} \end{cases}$	[15, 57, 106]
K_4	$\begin{cases} 17^{k/2}+1 & k \text{ even,} \\ 3 \cdot 17^{(k-1)/2}+1 & k \text{ odd} \end{cases}$	[166]
$sK_3, (k-s)C_4$	$\begin{cases} (k-s+3) \cdot 2 \cdot 5^{(s-1)/2}+1 \\ \quad \text{if } s \text{ is odd and } k-s>1, \\ (k-s+3) \cdot 5^{s/2}+1 \\ \quad \text{if } s \text{ is even and } k-s>1, \\ 6 \cdot 5^{(s-1)/2}+1 \\ \quad \text{if } s \text{ is odd and } k-s=1, \\ 3 \cdot 5^{s/2}+1 \\ \quad \text{if } s \text{ is even and } k-s=1 \end{cases}$	[224]
$B_m \quad (2 \le m \le 5)$	$\begin{cases} m+2 & \text{if } k=1, \\ (\mathrm{R}(B_m,B_m)-1) \cdot 5^{(k-2)/2}+1 & \text{if } k \text{ is even,} \\ 2 \cdot (\mathrm{R}(B_m,B_m)-1) \cdot 5^{(k-3)/2}+1 & \text{otherwise} \end{cases}$	[235]

The graph S_t^r is obtained from a star of order t by adding an extra r independent edges between the leaves of the so that there are r triangles and $t-2r-1$ pendent edges in S_t^r. For $r=0$ we obtain $S_t^r = K_{1,t-1}$, which are called *stars*. For $r = \frac{t-1}{2}$, if t is odd we obtain $S_t^r = F_{\frac{t-1}{2}}$, which are called *fans*.

Mao, Wang, Maganant, and Schiermeyer [178] found the following.

Theorem 4.32 ([178]) (1) *For* $k \ge 1$,

$$\mathrm{gr}_k(K_3; S_6^2) = \begin{cases} 2 \times 5^{\frac{k}{2}} + \frac{1}{4} \times 5^{\frac{k-2}{2}} + \frac{3}{4}, & \text{if } k \textit{ is even;} \\ \lceil \frac{51}{10} \times 5^{\frac{k-1}{2}} + \frac{1}{2} \rceil, & \text{if } k \textit{ is odd.} \end{cases}$$

(2) *For* $k \geq 3$,

$$\operatorname{gr}_k(K_3; S_8^2) = \begin{cases} 14 \times 5^{\frac{k-2}{2}} + \frac{1}{2} \times 5^{\frac{k-4}{2}} + \frac{1}{2}, & \text{if } k \text{ is even;} \\ 7 \times 5^{\frac{k-1}{2}} + \frac{1}{4} \times 5^{\frac{k-3}{2}} + \frac{3}{4}, & \text{if } k \text{ is odd.} \end{cases}$$

(3) *For* $k \geq 1$ *and* $t \geq 6$,

$$\begin{cases} 2(t-1) \times 5^{\frac{k-2}{2}} + 1 \leq \operatorname{gr}_k(K_3; S_t^2) \leq 2t \times 5^{\frac{k-2}{2}}, & \text{if } k \text{ is even;} \\ (t-1) \times 5^{\frac{k-1}{2}} + 1 \leq \operatorname{gr}_k(K_3; S_t^2) \leq t \times 5^{\frac{k-1}{2}}, & \text{if } k \text{ is odd.} \end{cases}$$

Finally, they also provided general bounds on the Gallai-Ramsey numbers for S_t^r.

Theorem 4.33 ([178]) *For* $t \geq 6r - 5$,

$$\begin{cases} 2(t-1) \times 5^{\frac{k-2}{2}} + 1 \leq \operatorname{gr}_k(K_3; S_t^r) \leq [2t + 8(r-1)] \times 5^{\frac{k-2}{2}} - 4(r-1), & \text{if } k \text{ is even;} \\ (t-1) \times 5^{\frac{k-1}{2}} + 1 \leq \operatorname{gr}_k(K_3; S_t^r) \leq [t + 4(r-1)] \times 5^{\frac{k-1}{2}} - 4(r-1), & \text{if } k \text{ is odd.} \end{cases}$$

Let S_t denote the star with t total vertices (and $t-1$ edges). Then for $t \geq 3$, let S_t^+ denote graph consisting of the star S_t with the addition of an edge between two of the pendant vertices, forming a triangle. Let P_t denote the path of order t. Then for $t \geq 3$, let P_t^+ denote the graph consisting of the path P_t with the addition of an edge between one end and the vertex at distance 2 along the path from that end, forming a triangle.

Wang, Mao, Zou, and Maganant [219] got the following results.

Theorem 4.34 ([219]) *For* $k \geq 1$ *and* $t \geq 3$,

$$\operatorname{gr}_k(K_3 : S_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 4.35 ([219]) *For* $t \geq 4$ *and* $k \geq 1$,

$$\operatorname{gr}_k(K_3 : P_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

In keeping with the trend of studying monochromatic subgraphs in Gallai colorings, we consider the fan graphs in this work. The *fan* graph with n triangles is denoted by F_n, where $F_n = K_1 + n\overline{K_2}$. Note that $F_1 = K_3$ and F_2 is a graph obtained from two triangles by sharing one vertex, often called a "bowtie". The main results of this work, the precise result for F_2, nearly sharp bounds for F_3, and general bounds for F_n, are contained in the following three theorems. First our sharp result for F_2.

Theorem 4.36 [180] *For* $k \geq 2$,

$$\operatorname{gr}_k(K_3; F_2) = \begin{cases} 9, & \text{if } k = 2; \\ \frac{83}{2} \cdot 5^{\frac{k-4}{2}} + \frac{1}{2}, & \text{if } k \text{ is even, } k \geq 4; \\ 4 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Next their general bounds (and sharp result for any even number of colors) for F_3.

Theorem 4.37 ([180]) *For $k \geq 2$,*

$$\begin{cases} \mathrm{gr}_k(K_3; F_3) = 14 \cdot 5^{\frac{k-2}{2}} - 1, & \text{if } k \text{ is even;} \\ \mathrm{gr}_k(K_3; F_3) = 33 \cdot 5^{\frac{k-3}{2}}, & \text{if } k = 3, 5; \\ 33 \cdot 5^{\frac{k-3}{2}} \leq \mathrm{gr}_k(K_3; F_3) \leq 33 \cdot 5^{\frac{k-3}{2}} + \frac{3}{4} \cdot 5^{\frac{k-5}{2}} - \frac{3}{4}, & \text{if } k \text{ is odd}, \ k \geq 7. \end{cases}$$

In particular, they conjectured the following, which claims that the lower bound in Theorem 4.37 is the sharp result.

Conjecture 8 ([180]) *For $k \geq 2$,*

$$\mathrm{gr}_k(K_3; F_3) = \begin{cases} 14 \cdot 5^{\frac{k-2}{2}} - 1, & \text{if } k \text{ is even;} \\ 33 \cdot 5^{\frac{k-3}{2}}, & \text{if } k \text{ is odd.} \end{cases}$$

Finally they got the general bound for all fans.

Theorem 4.38 ([180]) *For $k \geq 2$,*

$$\begin{cases} 4n \cdot 5^{\frac{k-2}{2}} + 1 \leq \mathrm{gr}_k(K_3; F_n) \leq 10n \cdot 5^{\frac{k-2}{2}} - \frac{5}{2}n + 1, & \text{if } k \text{ is even;} \\ 2n \cdot 5^{\frac{k-1}{2}} + 1 \leq \mathrm{gr}_k(K_3; F_n) \leq \frac{9}{2}n \cdot 5^{\frac{k-1}{2}} - \frac{5}{2}n + 1, & \text{if } k \text{ is odd.} \end{cases}$$

Let W_n be a wheel of order n, that is, $W_n = K_1 \vee C_{n-1}$ where C_{n-1} is the cycle on $n-1$ vertices. Mao, Wang, Maganant, and Schiermeyer [179] and Song, Wei, Zhang, Zhao [213] obtained the exact value of the Gallai-Ramsey number for W_5.

Theorem 4.39 ([179, 213]) *For $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : W_5) = \begin{cases} 5 & \text{if } k = 1, \\ 14 \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ 28 \cdot 5^{\frac{k-3}{2}} + 1 & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

We also obtained general upper and lower bounds.

Theorem 4.40 ([179]) *For $k \geq 2$ and $n \geq 6$, we have*

$$\mathrm{gr}_k(K_3 : W_n) \geq \begin{cases} (3n-4)5^{\frac{k-2}{2}} + 1 & \text{if } n \text{ is even and } k \text{ is even;} \\ (6n-8)5^{\frac{k-3}{2}} + 1 & \text{if } n \text{ is even and } k \text{ is odd;} \\ (2n-3)5^{\frac{k-2}{2}} + 1 & \text{if } n \text{ is odd and } k \text{ is even;} \\ (4n-6)5^{\frac{k-3}{2}} + 1 & \text{if } n \text{ is odd and } k \text{ is odd.} \end{cases}$$

Theorem 4.41 ([179]) *For $k \geq 3$ and $n \geq 6$, we have*

$$\mathrm{gr}_k(K_3 : W_n) \leq (n-4)^2 \cdot 30^k + k(n-1).$$

Let S_t denote the star with t total vertices (and $t-1$ edges). Then for $t \geq 3$, let S_t^+ denote graph consisting of the star S_t with the addition of an edge between two of the pendant vertices, forming a triangle. Note that $S_3^+ = K_3$.

Theorem 4.42 ([219]) *For $k \geq 1$ and $t \geq 3$,*

$$\operatorname{gr}_k(K_3 : S_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Let P_t denote the path of order t. Then for $t \geq 3$, let P_t^+ denote the graph consisting of the path P_t with the addition of an edge between one end and the vertex at distance 2 along the path from that end, forming a triangle. Note that $P_3^+ = K_3$ and $P_4^+ = S_4^+$.

Theorem 4.43 ([219]) *For $t \geq 4$ and $k \geq 1$,*

$$\operatorname{gr}_k(K_3 : P_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The *double star* $S(n,m)$ for integers $n \geq m \geq 0$ is the graph obtained from the union of two stars $K_{1,n}$ and $K_{1,m}$ by adding the edge between their centers. In this work, we prove bounds on the Gallai-Ramsey number of all double stars, along with sharp results for several small double stars.

One of the tools is the following by Grossman, Harary, and Klawe [100], who obtained the exact value of classical Ramsey number of double stars in most cases.

Theorem 4.44 ([100])

$$\mathrm{R}(S(n,m), S(n,m)) = \begin{cases} \max\{2n+1, n+2m+2\} & \text{if } n \text{ is odd and } m \leq 2 \\ \max\{2n+2, n+2m+2\} & \text{if } n \text{ is even or } m \geq 3, \\ & \text{and } n \leq \sqrt{2}m \text{ or } n \geq 3m. \end{cases}$$

This result was recently extended by Norin, Sun and Zhao [196] in the following result.

Theorem 4.45 ([196]) *For positive integers m and n with $m \leq n \leq 1.699(m+1)$,*

$$\mathrm{R}(S(n,m), S(n,m)) \leq n + 2m + 2.$$

Katona, Magnant, Mao, and Wang obtained the exact value of the Gallai Ramsey number of double stars $S(n,m)$ under an additional assumption that $n \geq 6m+7$.

Theorem 4.46 ([141]) *Let n, m, k be three integers with $m \geq 1$, $k \geq 3$ and $n \geq 6m+7$. Then*

$$\operatorname{gr}_k(K_3 : S(n,m)) = \begin{cases} 5 \cdot \frac{n}{2} + m(k-3) + 1 & \text{if } n \text{ is even,} \\ 5 \cdot \frac{n-1}{2} + m(k-3) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Next, they obtained the upper and lower bounds for Gallai Ramsey number of all double stars.

Theorem 4.47 ([141]) *Let n, m, k be three integers with $1 \leq m \leq n \leq 6m+6$, $k \geq 3$. Then*

$$\mathrm{gr}_k(K_3 : S(n,m)) \leq \begin{cases} \max\{5 \cdot \frac{n+2}{2}, 2n+6m+7\} + m(k-3) + 1 & \text{if } n \text{ is even,} \\ \max\{5 \cdot \frac{n+1}{2}, 2n+6m+7\} + m(k-3) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

and

$$\mathrm{gr}_k(K_3 : S(n,m)) \geq \begin{cases} \max\{5 \cdot \frac{n}{2}, n+3m+1\} + m(k-3) + 1 & \text{if } n \text{ is even,} \\ \max\{5 \cdot \frac{n-1}{2}, n+3m+1\} + m(k-3) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Chen, Li, and Pei [53] obtained the following result for complete bipartite graphs.

Theorem 4.48 ([53]) *For fixed integers $k \geq 2$ and $m \geq 1$, if n is large, then*

$$(1 - o(1))2^m n \leq \mathrm{gr}_k(K_3 : K_{m,n}) \leq (2m + 2^{m/2+1} + k)n + 4m^3.$$

Hamlin [116] studied the Gallai-Ramsey numbers for brooms.

Theorem 4.49 ([116]) *For all $k \geq 1$, we have*

$$\mathrm{gr}_k(K_3 : B_{2,5}) = 5 + 2k.$$

Theorem 4.50 ([116]) *If $m \geq \frac{7\ell}{2} + 3$, then*

$$2m + \ell - 2 + (k-2)\lceil \ell/2 \rceil \leq \mathrm{gr}_k(K_3 : B_{m,\ell}) \leq 3m - \lceil 3\ell/2 \rceil + (k-2)\lceil \ell/2 \rceil.$$

Hamlin [116] also listed sharp results for two classes of brooms and proposed a conjecture.

Theorem 4.51 ([116]) (1) *For $m \geq 2$ and $k \geq 2$ we have*

$$\mathrm{gr}_k(K_3 : B_{m,5}) = \begin{cases} m + 2k + 3 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k - 1 & \text{if } m \geq 4. \end{cases}$$

(2) *For $m \geq 2$ and $k \geq 2$ we have*

$$\mathrm{gr}_k(K_3 : B_{m,6}) = \begin{cases} m + 2k + 4 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k + 1 & \text{if } m \geq 4. \end{cases}$$

Conjecture 9 ([116]) *For any broom $B_{m,\ell}$ with $m, \ell \in \mathbb{Z}$ and $n = m + \ell$, and for any integer k with $k \geq 2$, we have*

$$\mathrm{gr}_k(K_3 : B_{m,\ell}) = \begin{cases} n + (k-1)(\lceil \ell/2 \rceil - 1) & \text{if } \ell \geq 2m - 1, \\ 2n - 3 + (k-4)(\lceil \ell/2 \rceil - 1) & \text{if } 4 \leq \ell \leq 2m - 2. \end{cases}$$

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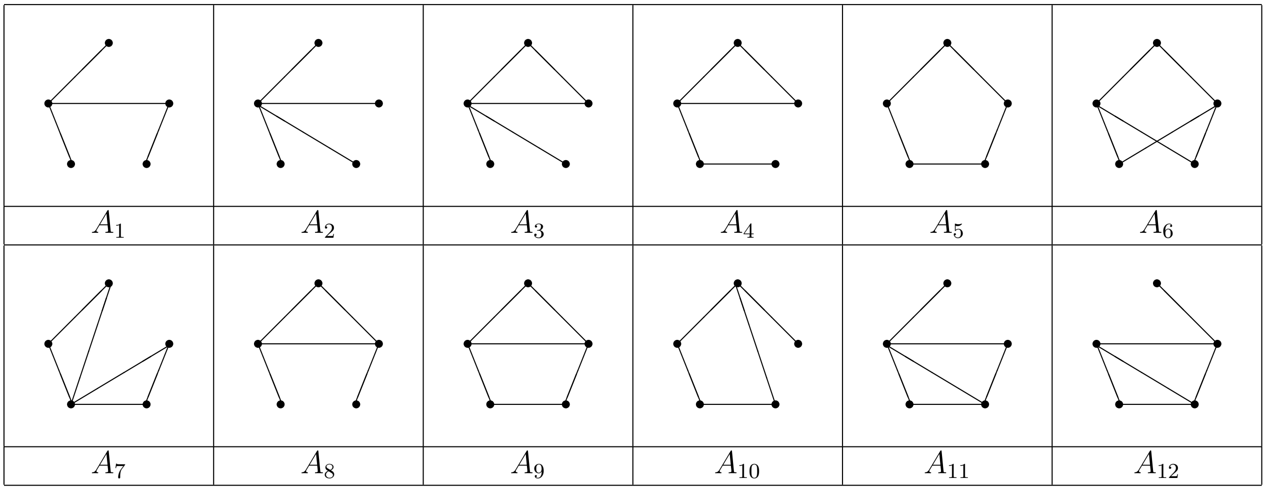

Table 1: Except the 4-path, all connected graphs with five vertices and at most six edges.

Li and Wang [160] investigated the Gallai-Ramsey numbers for connected graphs with five vertices and at most six edges.

Theorem 4.52 ([160]) (1) *For any integer $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : A_8) = \mathrm{gr}_k(K_3 : A_9) = \begin{cases} 8 \cdot 5^{(k-2)/2} + 1, & \text{if } k \text{ is even,} \\ 4 \cdot 5^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

(2) *For any integer $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : A_{11}) = \mathrm{gr}_k(K_3 : A_{12}) = \begin{cases} 9 \cdot 5^{(k-2)/2} + 1, & \text{if } k \text{ is even,} \\ 4 \cdot 5^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

For $n \geq 3$, let $A_{2,n}$ denote the graph obtained by adding $n-2$ pendent edges to a single vertex of C_4. Note that $A_{2,3}$ is the graph A_{10} in Table 1.

Theorem 4.53 ([160]) (1) *For all integers $k \geq 1$ and $n \in \{3,4\}$, $\mathrm{gr}_k(K_3 : A_{2,n}) = r_2(A_{2,n}) + k - 2$;*

(2) *For any integer $k \geq 3$, $\mathrm{gr}_k(K_3 : A_{2,5}) = k + 9$;*

(3) *For all integers $k \geq 3$ and $n \geq 6$,*

$$k(n-1) + 2 \geq \mathrm{gr}_k(K_3 : A_{2,n}) \geq \begin{cases} \frac{5n}{2} + k - 6, & \text{if } n \text{ is even,} \\ \frac{5n-1}{2} + k - 4, & \text{if } n \text{ is odd.} \end{cases}$$

A *kipas* $\widehat{K}_m$ is obtained by deleting one edge on the rim of a wheel with $m+1$ vertices. Zhao and Wei [233] obtained the following results.

Theorem 4.54 ([233]) *For all $k \geq 2$,*

$$\mathrm{gr}_k(K_3 : \widehat{K}_4) = \begin{cases} 2 \cdot 5^{k/2}, & if\ k\ is\ even, \\ 4 \cdot 5^{(k-1)/2} + 1. & if\ k\ is\ odd. \end{cases}$$

Conjecture 10 ([233]) *For all $k \geq 2$ and $m \geq 2$,*

$$\mathrm{gr}_k(K_3 : \widehat{K}_m) = \begin{cases} (\mathrm{R}_2(\widehat{K}_m) - 1) \cdot 5^{(k-2)/2} + 1, & if\ k\ is\ even\ and\ m\ is\ odd, \\ \mathrm{R}_2(\widehat{K}_m) + (m/2) \cdot (5^{k/2} - 5), & if\ k\ is\ even\ and\ m\ is\ even, \\ \max\{2(\mathrm{R}_2(\widehat{K}_m) - 1), 5m\} \cdot 5^{(k-3)/2} + 1, & if\ k \geq 3\ is\ odd. \end{cases}$$

Let $K_4 + e$ denote the graph on five vertices consisting of a K_4 with a pendant edge.

Theorem 4.55 ([156]) *For integers $k \geq 1$,*

$$\mathrm{gr}_k(K_3 : K_4 + e) = \mathrm{gr}'_k(K_3 : K_4 + e) = \begin{cases} 17^{k/2} + 1, & if\ k\ is\ even, \\ 4 \cdot 17^{(k-1)/2} + 1, & if\ k\ is\ odd. \end{cases}$$

For all $k \geq 2$ and $t_1 \geq t_2 \geq \cdots \geq t_k \geq 1$, Zhang, Zhu and Chen [229] obtained the following general lower and upper bounds for $\mathrm{gr}(t_1K_3, ..., t_kK_3)$. These bounds are related to the value of $\mathrm{gr}_k(K_3 : K_3)$.

Theorem 4.56 ([229]) *For all $k \geq 2$ and $t_1 \geq t_2 \geq \cdots \geq t_k \geq 1$,*

$$\mathrm{gr}(t_1K_3, ..., t_kK_3) \leq \mathrm{gr}_k(K_3 : K_3) + 3\sum_{i=1}^{k}(t_i - 1)$$

and

$$\mathrm{gr}(t_1K_3, ..., t_kK_3) \geq \begin{cases} \mathrm{gr}_k(K_3) + 3\sum_{i=1}^{k}(t_i - 1), & if\ k\ is\ even, \\ \mathrm{gr}_k(K_3) + 3\sum_{i=1}^{k}(t_i - 1), & if\ k\ is\ odd. \end{cases}$$

Taking $t_{k-1} = t_k = 1$ when k is odd, the upper bound is equal to the lower bound in Theorem 4.58. Thus we can obtain the following corollary.

Corollary 4.57 ([229]) *Let $k \geq 3$ be an odd integer. Then for all integers $t_1 \geq t_2 \geq \cdots \geq t_{k-1} \geq 1$ and $t_{k-1} = t_k = 1$,*

$$\mathrm{gr}(t_1K_3, ..., t_kK_3) \leq \mathrm{gr}_k(K_3) + 3\sum_{i=1}^{k}(t_i - 1).$$

Let $\mathrm{gr}_k(s; k - s) = GR(t_1K_3, ..., t_kK_3)$ when $t_1 = \cdots = t_s = 2$ and $t_{s+1} = \cdots = t_k = 1$.

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Theorem 4.58 ([229]) *Let $k \geq 2$ and $k \geq s \geq 0$ be integers.*
(a) If k is even, then $\mathrm{gr}_k(s; k-s) = 5^{k/2} + 2s + 1$.
(b) If k is odd, then

$$\mathrm{gr}_k(s; k-s) = \begin{cases} 2 \cdot 5^{(k-2)/2} + 3s + 1 & \text{if } s \leq k-2, \\ 2 \cdot 5^{(k-2)/2} + 3s & \text{if } s = k-1, \\ 2 \cdot 5^{(k-2)/2} + 3s - 1 & \text{if } s = k. \end{cases}$$

Wu, Magnant, Nowbandegani, and Xia [225] studied the Gallai-Ramsey numbers for bipartite graphs.

Theorem 4.59 ([225]) *Given a bipartite graph H and a positive integer R with $R \geq \max\{R(H,H), 3b(H) - 2\}$, if every Gallai-coloring of K_R using 3 colors, in which all parts of a Gallai-partition have order at most $s(H)-1$, contains a monochromatic copy of H, then*

$$\mathrm{gr}_k(K_3 : H) \leq R + (s(H) - 1)(k-2).$$

Chen, Li, and Pei [53] derived the following bounds for complete bipartite graphs.

Theorem 4.60 ([53]) *For fixed integers $k \geq 2$ and $m \geq 1$, if n is large, then*

$$(1 - o(1))2^m n \leq \mathrm{gr}_k(K_3 : K_{m,n}) \leq (2m + 2^{m/2+1} + k)n + 4m^3.$$

4.3 General Gallai-Ramsey Theory for Rainbow Trees

Thomason and Wagner [217] obtained the structural theorems for a rainbow P_i $(i = 4, 5)$.

Theorem 4.61 ([217]) *Let K_n, $n \geq 4$, be edge colored so that it contains no rainbow 3-path P_4. Then one of the following holds:*

(a) at most two colors are used;

(b) $n = 4$ and three colors are used, each color forming a 1-factor.

In an edge-colored graph, define $V^{(j)}$ as the set of vertices with at least one incident edge in color j and denote $E^{(j)}$ to be the set of edges of color j for a given color j.

Theorem 4.62 ([217]) *Let K_n, $n \geq 5$, be edge colored so that it contains no rainbow 4-path P_5. Then, after renumbering the colors, one of the following must hold:*

(a) at most three colors are used;

(b) color 1 is dominant, meaning that the sets V_j, $j \geq 2$, are disjoint;

(c) $K_n - a$ is monochromatic for some vertex a;

(d) there are three vertices a, b, c such that $E^{(2)} = \{ab\}$, $E^{(3)} = \{ac\}$, $E^{(4)}$ contains bc plus perhaps some edges incident with a, and every other edge is in $E^{(1)}$;

(e) there are four vertices a, b, c, d *such that* $\{ab\} \subseteq E^{(2)} \subseteq \{ab, cd\}$, $E^{(3)} = \{ac, bd\}$, $E^{(4)} = \{ad, bc\}$ *and every other edge is in* E_1*;*

(f) $n = 5$, $V(K_n) = \{a, b, c, d, e\}$, $E_1 = \{ad, ae, bc\}$, $E^{(2)} = \{bd, be, ac\}$, $E^{(3)} = \{cd, ce, ab\}$ *and* $E^{(4)} = \{de\}$.

Zou, Wang, Lai and Mao [236] obtained the following general results.

Theorem 4.63 ([236]) *For two integers* k, t *with* $k \geq 7$ *and* $k \geq t+1$*, if* H *is a graph of order* t*, then*

$$\mathrm{gr}_k(P_5 : H) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

Theorem 4.64 ([236]) *Let* k, t *be two integers with* $k = 5, 6$, $k \geq t+1$ *and* $t \geq 3$*. If* H *is a graph of order* t*, then* $\mathrm{gr}_k(P_5 : H) = 5$*.*

Theorem 4.65 ([236]) *Let* k, t *be two integers with* $k \geq 5$ *and* $k = t$*. If* H *is not a complete graph of order* t*, then*

$$\mathrm{gr}_k(P_5 : H) = t+1.$$

Theorem 4.66 ([236]) *For two integers* k, t *with* $k \geq 5$ *and* $k = t$, $\mathrm{gr}_k(P_5 : K_t) = (t-1)^2 + 1$*.*

From Theorems 4.63, 4.65 and 4.66, we obtain the following corollary.

Corollary 4.67 ([236]) *For integers* $k \geq 5$ *and* $k \geq t$*,*

$$\mathrm{gr}_k(P_5 : H) = \begin{cases} \max\{\lceil \frac{1+\sqrt{1+8k}}{2} \rceil, 5\}, & k \geq t+1; \\ \max\{\lceil \frac{1+\sqrt{1+8k}}{2} \rceil, t+1\}, & k = t \text{ and } H \text{ is not a complete graph}; \\ (t-1)^2 + 1, & k = t \text{ and } H \text{ is a complete graph}. \end{cases}$$

The following theorem shows the result on the graph H obtained from a complete graph K_t by deleting a maximally matching M.

Theorem 4.68 ([236]) *For two integers* k, t *with* $\lceil \frac{t+2}{2} \rceil \leq k \leq t-1$ *and* $k \geq 5$*, if* H *is a graph obtained from a complete graph* K_t *by deleting a maximally matching* M*, then*

$$\mathrm{gr}_k(P_5 : H) = \max\left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, t+1 \right\}.$$

The lower and upper bounds of $\mathrm{gr}_k(P_5 : H)$ on $\Delta(H)$ is shown in the following theorem.

Theorem 4.69 ([236]) *Let* k, t, p, q *be four positive integers with* $5 \leq k \leq t-1$*. If* $\Delta(H) - 1 = p(k-2) + q, q \in \{0, 1, \ldots, k-3\}$ *and* $\mathrm{R}_2(H) \geq t+1$*, then*

$$\max\{\Delta(H) + p, t+1\} \leq \mathrm{gr}_k(P_5 : H) \leq \mathrm{R}_2(H).$$

The graph S_t^r is obtained from a star of order t by adding an extra r independent edges between the leaves of the so that there are r triangles and $t-2r-1$ pendent edges in S_t^r. For $r=0$ we obtain $S_t^r = K_{1,t-1}$, which are called *stars*. For $r=\frac{t-1}{2}$, if t is odd we obtain $S_t^r = F_{\frac{t-1}{2}}$, which are called *fans*.

Theorem 4.70 ([236]) *Let k, r, t be three integers with $5 \leq k \leq t-1$ and $1 \leq r \leq k-2$. Then*

$$\mathrm{gr}_k(P_5 : S_t^r) = \max\{t+p-1, t+1\},$$

where $t-2 = p(k-2)+q$, $q \in \{0, 1, \ldots, k-3\}$.

Next, we show the result on the case $k=4$ for S_t^r.

Theorem 4.71 ([236]) *Let k, t, r be three integers with $k=4$, $t \geq 6$ and $r=1,2$. Then*

$$\mathrm{gr}_4(P_5 : S_t^r) = t+p-1,$$

where $t-2 = 2p+q$ and $q \in \{0,1\}$.

The following corollary is immediate.

Corollary 4.72 ([236]) *Let k, t, r be three integers with $k=4$, $t \geq 6$ and $r=1,2$. Then*

$$\mathrm{gr}_4(P_5 : S_t^r) = \begin{cases} 6, & t=4,5; \\ t+p-1, & t \geq 6. \end{cases}$$

Theorem 4.73 ([236]) *If $r \geq 3$ and t is odd, then*

$$\mathrm{gr}_4(P_5 : S_t^r) = \begin{cases} \frac{3t-5}{2}, & 3 \leq r \leq \lfloor \frac{t-1}{4} \rfloor; \\ t+2r-2, & \lceil \frac{t-1}{4} \rceil \leq r \leq \frac{t-3}{2}. \end{cases}$$

Theorem 4.74 ([236]) *Let k, r, t be three integers with $k=4$ and $r \geq 3$. If t is even, then*

$$\mathrm{gr}_4(P_5 : S_t^r) = \begin{cases} \frac{3t-4}{2}, & 3 \leq r \leq \lfloor \frac{t}{4} \rfloor; \\ t+2r-2, & \lceil \frac{t}{4} \rceil \leq r \leq \frac{t-2}{2}. \end{cases}$$

Finally, we show the result on S_4^1, S_5^1, S_6^1.

Theorem 4.75 ([236]) *For integers $k \geq 3$, we have*

$$\mathrm{gr}_k(P_5 : S_4^1) = \begin{cases} 17, & k=3; \\ 6, & k=4; \\ 5, & k=5,6; \\ \ell, & \binom{\ell-1}{2}+1 \leq k \leq \binom{\ell}{2} \text{ and } \ell \geq 5. \end{cases}$$

Theorem 4.76 ([236]) *For integers $k \geq 3$, we have*

$$\operatorname{gr}_k(P_5 : S_5^1) = \begin{cases} 21, & k = 3; \\ 6, & k = 4, 5; \\ 5, & k = 6; \\ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & k \geq 7. \end{cases}$$

Theorem 4.77 ([236]) *For integer $k \geq 3$, we have*

$$\operatorname{gr}_k(P_5 : S_6^1) = \begin{cases} 26, & k = 3; \\ 7, & 4 \leq k \leq 6; \\ \lceil \frac{1+\sqrt{1+8k}}{2} \rceil, & k \geq 7. \end{cases}$$

A *pineapple* $PA_{t,\omega}$ is a graph obtained from the complete graph K_ω by attaching $t - \omega$ pendent vertices to the same vertices of K_ω, we suppose that $t \geq \omega + 1$.

Theorem 4.78 ([236]) *Let k, t, ω be three integers with $k = \omega$ and $k \geq 4$. Then*

$$\operatorname{gr}_k(P_5 : PA_{t,\omega}) = (\omega - 1)(t - 1) + 1.$$

Theorem 4.79 ([236]) *Let k, t, ω be three integers with $4 \leq k \leq \omega - 1$ and $\omega = 5, 6$. Then*

$$\operatorname{gr}_k(P_5 : PA_{t,\omega}) = (\omega - 1)(t - 1) + 1.$$

Theorem 4.80 ([236]) *Let k, t, ω be three integers with $k = 4$ and $\omega \geq 7$. Then*

$$(\omega - 1)(t - 1) + 1 \leq \operatorname{gr}_4(P_5 : PA_{t,\omega}) \leq 3\,\mathrm{R}_2(PA_{t,\omega}) - 2.$$

Sah [203] obtained the following result.

Theorem 4.81 ([203]) *There is an absolute constant $c > 0$ such that for $k \geq 3$,*

$$\mathrm{R}_2(k+1) \leq \binom{2k}{k} e^{-c(\log k)^2}.$$

By the upper bound in Theorem 4.81, they derived the following result.

Theorem 4.82 ([236]) *There is an absolute constant $c > 0$ such that for $\omega \geq 4$,*

$$\mathrm{R}_2(PA_{t,\omega}) \leq \binom{2\omega - 2}{\omega - 1} e^{-c\log^2(\omega - 1)} + (t - 2)(\omega - 1).$$

The following corollaries are immediate.

Corollary 4.83 ([236]) *Let k, t, ω be three integers with $k = 4$ and $\omega \geq 7$. Then*

$$(\omega - 1)(t - 1) + 1 \leq \operatorname{gr}_4(P_5 : PA_{t,\omega}) \leq 3\binom{2\omega - 2}{\omega - 1} e^{-c\log^2(\omega - 1)} + (t - 2)(\omega - 1) - 2,$$

where $c > 0$ is an absolute constant.

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Corollary 4.84 ([236]) *Let k, t, ω be three integers with $5 \leq k \leq \omega - 1$ and $\omega \geq 7$. Then*

$$(\omega - 1)(t - 1) + 1 \leq \mathrm{gr}_k(P_5 : PA_{t,\omega}) \leq \binom{2\omega - 2}{\omega - 1} e^{-c \log^2(\omega-1)} + (t - 2)(\omega - 1).$$

Wei, He, Mao and Zhou [221] obtained the following results for fans and wheels.

Theorem 4.85 ([221]) (1) *For $k \geq 3$,*

k	3	4	5	6	≥ 7
$\mathrm{gr}_k(P_5 : F_2)$	$\in [21, 32]$	7	6	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

(2) *For $k \geq 3$,*

k	3	4	5	6	7	≥ 10
$\mathrm{gr}_k(P_5 : F_4)$	$[21, 65]$	11	8	8	8	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

(3) *For $k \geq 3$,*

k	3	4	5	6	7	8	9	≥ 10
$\mathrm{gr}_k(P_5 : F_4)$	$[25, 89]$	15	10	10	10	10	10	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

Theorem 4.86 ([221]) *Let k be an integer with $k \geq 5$.*
(1)

$$\mathrm{gr}_3(P_5 : F_q) = \mathrm{R}_3(F_q) \in \begin{cases} [21, 32] & \text{if } q = 2, \\ [21, 65] & \text{if } q = 3, \\ [9q - 11, 24q - 7] & \text{if } 4 \leq q \leq 12, \\ [9q - 11, 45q/2 + 11] & \text{if } q \geq 13. \end{cases}$$

(2) *For all $q \geq 2$, $\mathrm{gr}_4(P_5 : F_q) \geq 4q - 1$.*
(3) $\mathrm{gr}_k(P_5 : F_q) = \lceil \frac{1+\sqrt{1+8k}}{2} \rceil$ *for* $\lceil \frac{1+\sqrt{1+8k}}{2} \rceil \geq 2q + 2$.

Theorem 4.87 ([221]) *For $k \geq 3$,*

k	3	4	5	6	≥ 7
$\mathrm{gr}_k(P_5 : W_5)$	$\in [21, 608]$	7	6	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$
$\mathrm{gr}_k(P_5 : W_6)$	$\in [91, 959]$	16	7	7	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

Theorem 4.88 ([221]) *For $k \geq 3$,*

$$\mathrm{gr}_k(P_5 : W_7) = \begin{cases} \mathrm{R}_3(W_7) \in [31, 986] & \text{if } k = 3, \\ 11 & \text{if } k = 4, \\ 8 & \text{if } k = 5, 6, 7, \\ \lceil \frac{1+\sqrt{1+8k}}{2} \rceil & \text{if } k \geq 8. \end{cases}$$

Theorem 4.89 ([221]) *Let k be an integer with $k \geq 5$.*

(1) *For all $s \geq 4$,*

$$\mathrm{gr}_3(P_5 : W_s) = \mathrm{R}_3(W_s) \begin{cases} = 128 & \text{if } s = 4, \\ \in [21, 608] & \text{if } s = 5, \\ \in [91, 959] & \text{if } s = 6, \\ \in [31, 986] & \text{if } s = 7, \\ \in [127, 1607] & \text{if } s = 8, \\ \in [18s - 17, 189s - 635 + 27\sqrt{9(2s-7)^2 + 24}] & \text{if } s \geq 10 \text{ is even}, \\ \in [5s - 4, 243s - 554 + o(1)(s-1)] & \text{if } s \geq 9 \text{ is odd}. \end{cases}$$

(2) *For all $s \geq 4$,*

$$\mathrm{gr}_4(P_5 : W_s) \begin{cases} \geq 2s - 3 & \text{if } s \text{ is odd}, \\ = 3s - 2 & \text{if } 4 \leq s \leq 16 \text{ is even}, \\ \in [3s - 2, 4s - 18] & \text{if } s \geq 18 \text{ is even}, \end{cases}$$

and the equality holds for $s = 5$.

(3) $\mathrm{gr}_k(P_5 : W_s) = \lceil \frac{1+\sqrt{1+8k}}{2} \rceil$ *for* $\lceil \frac{1+\sqrt{1+8k}}{2} \rceil \geq s + 1$.

Wei, Mao, Schiermeyer, and Wang [222] got the following results.

Theorem 4.90 ([222]) *For $k \geq 3$,*

k	3	4	5	6	≥ 7
$\mathrm{gr}_k(P_5 : C_3)$	17	6	5	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$
$\mathrm{gr}_k(P_5 : C_4)$	11	6	5	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$
$\mathrm{gr}_k(P_5 : C_5)$	17	7	6	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

Theorem 4.91 ([222]) *Let k be an integer with $k \geq 5$.*

(1) *For $\ell \geq 3$, the exact values of $\mathrm{gr}_3(P_5 : C_\ell)$ are given in the following table.*

ℓ	3	4	5	6	7	8	*even ℓ is large enough*	*odd ℓ is large enough*
$\mathrm{gr}_3(P_5 : C_\ell)$	17	11	17	12	25	16	$(2 + o(1))\ell$	$(4 + o(1))\ell$

(2) $\mathrm{gr}_4(P_5 : C_3) = \mathrm{gr}_4(P_5 : C_4) = 6$, $\mathrm{gr}_4(P_5 : C_5) = \mathrm{gr}_4(P_5 : C_6) = 7$, $\mathrm{gr}_4(P_5 : C_7) = 8$, $\mathrm{gr}_4(P_5 : C_8) = 9$, $\mathrm{gr}_4(P_5 : C_9) = 13$ *and for $\ell \geq 10$,*

$$\mathrm{gr}_4(P_5 : C_\ell) = \begin{cases} \in [\frac{3\ell-3}{2}, \frac{3\ell-1}{2}] & \text{if } 11 \leq \ell \leq 19,\ \ell \text{ is odd}, \\ \in [\frac{3\ell-3}{2}, 2\ell - 11] & \text{if } \ell = 21, \\ \in [2\ell - 13, 2\ell - 11] & \text{if } \ell \geq 23,\ \ell \text{ is odd}, \\ = \frac{3\ell}{2} - 1 & \text{if } \ell \geq 10,\ \ell \text{ is even}. \end{cases}$$

(3) $\mathrm{gr}_k(P_5 : C_\ell) = \lceil \frac{1+\sqrt{1+8k}}{2} \rceil$ *for* $\lceil \frac{1+\sqrt{1+8k}}{2} \rceil \geq \ell + 1$.

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Theorem 4.92 ([222]) *For $k \geq 3$,*

k	3	4	5	6	≥ 7
$\mathrm{gr}_k(P_5 : B_2)$	$[28, 30]$	6	5	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$
$\mathrm{gr}_k(P_5 : B_3)$	$[28, 83]$	7	6	5	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$
$\mathrm{gr}_k(P_5 : B_4)$	$[35, 116]$	10	7	7	$\lceil \frac{1+\sqrt{1+8k}}{2} \rceil$

Theorem 4.93 ([222]) *Let k be an integer with $k \geq 5$.*

(1) *For any $m \geq 2$, $\mathrm{gr}_3(P_5 : B_m) = \mathrm{R}_3(B_m)$, where the bounds of $\mathrm{R}_3(B_m)$ are as follows.*

m	2	3	4	*m is large enough*
$\mathrm{R}_3(B_m)$	$[28, 30]$	$[28, 83]$	$[35, 116]$	$[2(4 + o(1))m - 1, 33m - 16]$

(2) $\mathrm{gr}_4(P_5 : B_m) = \begin{cases} 6 & \textit{if } m = 2, \\ 3m - 2 & \textit{if } m \geq 3. \end{cases}$

(3) $\mathrm{gr}_k(P_5 : B_m) = \lceil \frac{1+\sqrt{1+8k}}{2} \rceil$ *for* $\lceil \frac{1+\sqrt{1+8k}}{2} \rceil \geq m + 3$.

Li, Wang and Liu [159] obtained the following results.

Proposition 6 ([159]) *For $k \geq 4$ and $n \geq 5$, let G be a k-coloring (using all k colors) of K_n with no rainbow P_5. If $k = 4$, then $n \geq 5$, and if $k \geq 5$, then $n \geq k$.*

Proposition 7 ([159]) *For $5 \leq n \leq k - 1$ and $\binom{n}{2} \geq k$, there is a rainbow P_5 in any k-coloring (using all k colors) of K_n.*

Theorem 4.94 ([159]) (1) $\mathrm{gr}_4(P_5 : K_3) = 6$; $\mathrm{gr}_4(P_5 : K_4) = 10$; $\mathrm{gr}_4(P_5 : K_5) = 22$;

(2) *For $t \geq 3$, $k \geq \max\{t + 1, 5\}$, $\binom{n}{2} \geq k$ and $n \geq 5$, there is a monochromatic K_t in any rainbow P_5-free coloring (using all k colors) of complete graph K_n.*

Theorem 4.95 ([159]) (1) *For $t \geq 5$, $\mathrm{gr}_t(P_5 : K_t) = t^2 - 2t + 2$.*

(2) *For $t \geq 5$, $\mathrm{gr}_{t-1}(P_5 : K_t) = t^2 - 4t + 3 + R(3, t)$.*

Theorem 4.96 ([159]) *For $t \geq 6$ and $4 \leq k \leq t - 2$, $(t - 1)(k - 2) + R(t - k + 2, t) \leq \mathrm{gr}_k(P_5 : K_t) \leq t - 2k + 2 + (k - 1)R(t, t)$.*

Li, Besse, Magnant, Wang, and Watts [155] got the following results.

Theorem 4.97 ([155]) *For any integer $k \geq 2$ and any graph H with no isolated vertices, we have*

$$\mathrm{gr}'_k(P_4 : H) = \mathrm{R}_2(H)$$

except when $H = P_3$ and $k \geq 3$, in which case $\mathrm{gr}'_k(P_4 : P_3) = 5$.

They proposed the following conjecture.

Conjecture 11 ([155]) *For any integer $k \geq 3$ and any graph H with no isolated vertices, we have*

$$\mathrm{gr}'_k(P_5 : H) = \mathrm{R}_3(H).$$

Theorem 4.98 ([155]) *Conjecture 11 holds for all connected graphs and bipartite graphs.*

Zhou, Li, Mao, and Wei derived the following results.

Theorem 4.99 ([237]) *For integers $q \geq 3$ and $k \geq 4$, we have*

$$\mathrm{gr}_k(K_{1,3} : P_3 \cup K_{1,q}) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & 3 \leq q \leq 2k-6; \\ q+4, & 2k-5 \leq q \leq 4k-9; \\ q + \lfloor \frac{q}{k-2} \rfloor + 1, & q \geq 4k-8. \end{cases}$$

Theorem 4.100 ([237]) *For integer $q \geqslant 3$ and $k \geqslant 4$, we have*

$$\mathrm{gr}_k(P_5 : P_3 \cup K_{1,q}) = \begin{cases} \lceil \frac{1+\sqrt{1+8k}}{2} \rceil, & 3 \leqslant q \leqslant k-5; \\ q+5, & k-4 \leqslant q \leqslant 4k-9; \\ q + \lfloor \frac{q}{k-2} \rfloor + 1, & q \geqslant 4k-8. \end{cases}$$

Theorem 4.101 ([237]) *For $q \geq 3$, we have*

$$\mathrm{gr}_4(P_4^+ : P_3 \cup K_{1,q}) = \begin{cases} q+6, & 3 \leq q \leq 9; \\ \lceil \frac{3q+1}{2} \rceil, & q \geq 10. \end{cases}$$

Corollary 4.102 ([237]) *For $k \geq 5$, we have*

$$\mathrm{gr}_k(P_4^+ : P_3 \cup K_{1,q}) = \begin{cases} \lceil \frac{1+\sqrt{1+8k}}{2} \rceil, & 3 \leq q \leq 2k-6; \\ q+4, & 2k-5 \leq q \leq 4k-9; \\ q + \lfloor \frac{q}{k-2} \rfloor + 1, & q \geq 4k-8. \end{cases}$$

This problem was also considered in [120]. Using different notation, the authors of [83] consider the cases $\mathrm{gr}_k(P_4, K_{1,3})$, $\mathrm{gr}_k(K_{1,3}, P_4)$, $\mathrm{gr}_k(P_4, P_4)$, $\mathrm{gr}_k(K_{1,3}, K_3)$, $\mathrm{gr}_k(P_4, K_3)$, $\mathrm{gr}_k(K_3, P_4)$, and show the following result.

Theorem 4.103 ([83]) *For $m \leq k$,*

$$\mathrm{gr}'_k(K_{1,m}, K_{1,n}) = (n-1)(m-1) + 2.$$

Mao [176] obtained the following result.

Theorem 4.104 ([176]) *Let k, s, t be two integers with $k \geq 2$ and $t \geq 3$. Let $x = (s - i)(t-1) + i$, where $2 \leq i \leq s-1$. Then*

(1)

$$\mathrm{gr}_k(K_{1,s} : K_{1,t}) = \begin{cases} k(t-1)+2 & \text{if } \textit{one of } t,k \textit{ is odd and } s > k, \\ k(t-1)+1 & \text{if } t,k \textit{ are even and } s > k, \\ a & \text{if } k = \binom{a}{2} - y \textit{ and } a \geq s+3 \textit{ and } 0 \leq y < a-1. \end{cases}$$

(2) *If $k = \binom{a}{2} - y \geq s$ and $a \leq s+2$ and $0 \leq y < a-1$, then*

$$(s-1)(t-1)+2 \geq \mathrm{gr}_k(K_{1,s} : K_{1,t}) \geq \begin{cases} x & \text{if } x \textit{ is even}, \\ x-1 & \text{if } x \textit{ is odd}. \end{cases}$$

where $s - i + \frac{x}{2}(i-2) < k \leq s - i + \frac{x}{2}(i-1)$ if x is even, and $s - i + \frac{x-1}{2}(i-3) < k \leq s - i + \frac{x-1}{2}(i-2)$ if x is odd.

4.4 General Gallai-Ramsey Theory for Other Rainbow Graphs

In a similar work, Theorem 4.1 was extended by Fujita and Magnant as follows.

Theorem 4.105 ([87]) *In any rainbow S_3^+-free coloring G of a complete graph, one of the following holds:*

> *(a) $V(G)$ can be partitioned such that there are at most 2 colors on the edges between the parts; or*
>
> *(b) There are three (different colored) monochromatic spanning trees, and moreover, there exists a partition of $V(G)$ with exactly 3 colors on edges between parts and between each pair of parts, the edges have only one color.*

Each conclusion of this result is best possible. In general, the following slightly weaker result also holds.

Theorem 4.106 ([87]) *For $k \geq 4$, in any rainbow S_k^+-free coloring G of a complete graph, there exists a partition of $V(G)$ such that between the parts, there are at most k colors. Furthermore, there exists a coloring with k colors between parts.*

Using Theorems 4.5 and 4.105, the authors also proved the following extension of Theorem 4.5. In light of Theorem 4.5, one may be inclined to ask whether there exists a monochromatic spanning broom in a rainbow S_3^+-free coloring. Unfortunately, this is not the case by the following example. Consider $G = G_1 \cup G_2 \cup G_3 \cup G_4$ where each G_i is a complete graph with all edges colored with color 1. The edges $E(G_1, G_2)$ and $E(G_3, G_4)$ are also colored with color 1 while the edges of $E(G_1, G_3) \cup E(G_2, G_4)$ have color 2 and $E(G_1, G_4) \cup E(G_2, G_3)$ have color 3 (where $E(A, B)$ denotes the set of all edges between A and B). This coloring contains no rainbow S_3^+ and no monochromatic spanning structure.

Theorem 4.107 ([87]) *In any rainbow S_3^+-free coloring of a complete graph, there exists a spanning 2 colored broom.*

Extending Theorem 4.17, Fujita and Magnant proved the following collection of results. The proof of Item 4.108 uses the decomposition from Theorem 4.105 whereas the proofs of Items 4.108 and 4.108 use techniques similar to those of Theorem 4.11 and Item 4.17 in Theorem 4.17, respectively.

Theorem 4.108 ([87]) *For all $k \geq 1$,*

(1) $\mathrm{gr}_k(S_3^+ : P_4) = k + 3$.

(2) $\mathrm{gr}_k(S_3^+ : K_3) = \lambda(k)$ *where* $\lambda(k) = 5^{k/2} + 1$ *for k even and* $\lambda(k) = 2 \cdot 5^{(k-1)/2} + 1$ *for k odd.*

(3) $\mathrm{gr}_k(S_3^+ : C_4) = k + 4$.

As an extension of Theorem 4.16, Fujita and Magnant also proved the following for rainbow S_3^+-free colorings.

Theorem 4.109 ([87]) *Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $\mathrm{gr}_k(S_3^+, H)$ is exponential in k. If H is bipartite, then $\mathrm{gr}_k(S_3^+, H)$ is linear in k.*

Note that the results of Theorems 4.108 and 4.109 provide the same numbers as the rainbow triangle free cases but the proofs are more complicated due to the weaker structure from Theorem 4.105.

Li and Wang [162] obtained the following results.

Theorem 4.110 ([162]) (1) *For any integer $k \geq 5$, $\mathrm{gr}_k(S_3^+ : P_5) = \mathrm{gr}'_k(S_3^+ : P_5) = k + 4$.*
(2) *For any integer $k \geq 1$, $\mathrm{gr}_k(S_3^+ : P_5) = \mathrm{gr}'_k(S_3^+ : 2P_3) = k + 5$.*

Theorem 4.111 ([162]) *For integers $m \geq 1$ and $k \geq 1$,*

$$\mathrm{gr}_k(S_3^+ : P_5) = \mathrm{gr}'_k(S_3^+ : mP_2) = (m-1)k + m + 1.$$

Corollary 4.112 ([162]) *For any integer $k \geq 5$, $\mathrm{gr}_k(S_3^+ : P_5) = \mathrm{gr}'_k(S_3^+ : P_3 \cup P_2) = k + 4$.*

When we consider monochromatic stars, the picture becomes far more complicated.

Conjecture 12 ([87]) *For all $k \geq 4$,*

$$\mathrm{gr}_k(S_3^+ : S_t) = 3t - 2k + 4.$$

This conjecture would be sharp by the following example. Given an integer k, let $G = A_1 \cup A_2 \cup A_3 \cup H$ where H is a rainbow triangle free coloring of a complete graph on $k-2$ vertices where using colors $4, \dots, k$, and A_i is a complete graph of order $\frac{n-k+2}{3}$ colored entirely with color i for each $i = 1, 2, 3$. The edges of $E(A_1, A_2)$ have color 3, $E(A_2, A_3)$ have color 1 and $E(A_1, A_3)$ have color 2. Also, $E(H, A_1)$ have color 3, $E(H, A_2)$ have color 1 and $E(H, A_3)$ have color 2. The graph G contains no rainbow S_3^+ but contains a star of order $\frac{n+2k-4}{3}$.

Fortunately, the following result shows that if we allow the use of fewer colors on the edges, then things become much easier.

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Theorem 4.113 ([87]) *For all $t \geq 1$, and $k \geq 3$, we have* $\mathrm{gr}'_k(S_3^+ : S_t) = 3t - 1$.

Li and Wang [161] disproved Conjecture 12, and they obtained the following results.

Theorem 4.114 ([161]) *For $t \geq 1$, $k \geq 3t - 2$ and $n \geq 4$, there is either a rainbow S_3^+ or a monochromatic $K_{1,t}$ in any colored K_n using exactly k colors.*

Theorem 4.115 ([161]) *For $k \geq 4$ and $n \geq 4$, there is either a rainbow S_3^+ or a monochromatic $K_{1,3}$ in any colored K_n using exactly k colors.*

Theorem 4.116 ([161]) *The following statements hold:*

(1) $\mathrm{gr}_4(S_3^+ : K_{1,4}) = 9$;

(2) $\mathrm{gr}_5(S_3^+ : K_{1,4}) = 7$;

(3) *For $k \geq 6$ and $n \geq 4$, there is either a rainbow S_3^+ or a monochromatic $K_{1,4}$ in any colored K_n using exactly k colors.*

Theorem 4.117 ([161]) *The following statements hold:*

(1) *For $4 \leq k \leq 7$, we have* $\mathrm{gr}_k(S_3^+ : K_{1,5}) = 11$;

(2) *For $k \geq 8$ and $n \geq 5$, there is either a rainbow S_3^+ or a monochromatic $K_{1,5}$ in any colored K_n using exactly k colors.*

Conjecture 13 ([161]) *For $t \geq 5$ and $n \geq 4$, there is either a rainbow S_3^+ or a monochromatic $K_{1,t}$ in any colored K_n using exactly k colors, where $k \geq (5t-9)/2$ if t is odd, and $k \geq (5t-12)/2$ if t is even.*

Given a bipartite graph H, let $s(H)$ be the order of the smallest part and let $b(H)$ be the order of the biggest part of any bipartition of H.

4.5 Bipartite Gallai-Ramsey Theory

The complete bipartite graphs with s vertices in one partite set U and t vertices in the other partite set V is denoted by $K_{s,t}$.

Li, Wang and Liu [159] introduced the following concept.

Definition 8 ([159]) *The k-*colored bipartite Gallai-Ramsey number $\mathrm{bgr}_k(G : H)$ *as the minimum integer n such that $n^2 \geq k$ and for every $N \geq n$, every rainbow G-free coloring (using all k colors) of the complete bipartite graph $K_{N,N}$ contains a monochromatic copy of H.*

They give the structure of complete bipartite graphs without small rainbow paths.

Theorem 4.118 ([159]) *Let $K_{n,n} = G(U,V)$, $n \geq 2$, be edge colored such that it contains no rainbow P_4. Then one of the following holds (replacing U and V if necessary):*

(a) *at most two colors are used;*

(b) *U can be partitioned into k non-empty parts $U_1, U_2, \ldots, U_k$ such that $c(U_i, V) = i$, $i = 1, 2, \ldots, k$, where k is the number of colors used in the coloring.*

Theorem 4.119 ([159]) *Let $K_{n,n} = G(U, V)$, $n \geq 3$, be edge colored such that it contains no rainbow P_5. Then, after renumbering the colors, one of the following holds (replacing U and V if necessary):*

(a) *at most three colors are used;*

(b) *U can be partitioned into two parts U_1 and U_2 with $|U_1| \geq 1$, $|U2| \geq 0$, and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0$, $|V_j| \geq 1$, $j = 2, 3, \ldots, k$, such that $c(V_i, U_1) = i$, $c(V_i, U_2) = 1$, $i = 1, 2, \ldots, k$, where k is the number of colors used in the coloring;*

(c) *U can be partitioned into k parts $U_1, U_2, \ldots, U_k$ with $|U_1| \geq 0$, $|U_j| \geq 1$, $j = 2, 3, \ldots, k$, and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0$, $|V_j| \geq 1$, $j = 2, 3, \ldots, k$, such that only colors 1 and i can be used on the edges of $E(U_i, V_i)$, $i = 1, 2, \ldots, k$, and every other edge is in $E^{(1)}$, where k is the number of colors used in the coloring;*

(d) *$n = 3$, $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, $E^{(1)} = \{u_1v_1, u_2v_3, u_3v_2\}$, $E^{(2)} = \{u_1v_3, u_2v_1\}$, $E^{(3)} = \{u_1v_2, u_3v_1\}$ and $E^{(4)} = \{u_2v_2, u_3v_3\}$;*

(e) *$n = 4$, $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, $E^{(1)} = \{u_1v_1, u_2v_3, u_3v_2, u_4v_4\}$, $E^{(2)} = \{u_1v_3, u_2v_1, u_3v_4, u_4v_2\}$, $E^{(3)} = \{u_1v_2, u_2v_4, u_3v_1, u_4v_3\}$ and $E^{(4)} = \{u_1v_4, u_2v_2, u_3v_3, u_4v_1\}$.*

Proposition 8 ([159]) *For $n \geq 2$ and $k \geq 3$, let G be a k-coloring (using all k colors) of $K_{n,n}$ with no rainbow P_4, then $n \geq k$.*

Theorem 4.120 ([159]) (1) *In any rainbow P_4-free coloring (using at least three colors) of $K_{n,n}$, there is a monochromatic $K_{1,t}$ with $1 \leq t \leq n$.*

(2) *For $k \geq 3$ and $t \geq k$, there is either a rainbow P_4 or a monochromatic $K_{1,t}$ in any coloring (using all k colors) of $K_{t,t}$.*

Theorem 4.121 ([159]) *For $k \geq 3$ and $t \geq s \geq 2$,* $\mathrm{bgr}_k(P_4 : K_{s,t}) = \max\{t, (s-1)k + 1\}$.

Corollary 4.122 ([159]) *For $k \geq 3$, $t_1 \geq s_1 \geq 2$, and $t_2 \geq s_2 \geq 2$, if $s_1 \geq s_2$, $t_1 \geq t_2$, then* $\mathrm{bgr}_k(P_4 : K_{s_2,t_2}) \leq \mathrm{bgr}_k(P_4 : K_{s_1,t_1})$. *Moreover, the equality is attained if and only if one of the following holds:*

(1) *$t_1 \leq (s_1 - 1)k + 1$, $t_2 < (s_2 - 1)k + 1$, and $s_1 = s_2$;*

(2) *$t_1 \geq (s1 - 1)k + 1$, $t_2 \geq (s2 - 1)k + 1$, and $t_1 = t_2$.*

Corollary 4.123 ([159]) *For $k \geq 3$, $t_1 \geq s_1 \geq 2$, and $t_2 \geq s_2 \geq 2$, if $s_1 \geq s_2$, $t_1 < t_2$, then one of the following holds:*

(1) $\mathrm{bgr}_k(P_4 : K_{s_1,t_1}) < \mathrm{bgr}_k(P_4 : K_{s_2,t_2})$ *if and only if* $t_1 \geq (s_1-1)k+1$ *and* $t_2 \geq (s_2-1)k+1$, *or* $t_1 < (s_1-1)k+1 < t_2$;

(2) $\mathrm{bgr}_k(P_4 : K_{s_1,t_1}) > \mathrm{bgr}_k(P_4 : K_{s_2,t_2})$ *if and only if* $t_1 < (s_1-1)k+1$ *and* $(s_2-1)k+1 \leq t_2 < (s_1-1)k+1$, *or* $t_1 < (s_1-1)k+1$, $t_2 < (s_2-1)k+1$ *and* $s_1 > s_2$;

(3) $\mathrm{bgr}_k(P_4 : K_{s_1,t_1}) = \mathrm{bgr}_k(P_4 : K_{s_2,t_2})$ *if and only if* $t_1 < (s_1-1)k+1 = t_2$, *or* $t_1 < t_2 < (s_1-1)k+1 = (s_2-1)k+1$.

Theorem 4.124 ([159]) *For* $k \geq 3$ *and* $r \geq 2$, *if*

$$F \in \{C_{2r}, P_{2r}, P_{2r+1}, rK_2, P_r \cup P_{r+1}, 2P_r (r \text{ is even}), (r/2)C_4 (r \geq 4 \text{ is even}), 2C_r (r \geq 4 \text{ is even})\},$$

then $\mathrm{bgr}_k(P_4 : F) = (r-1)k+1$.

Proposition 9 ([159]) *If* $K_{n,n}$ *is edge colored (using* $k \geq 4$ *colors) such that there is no rainbow* P_5, *then* $n \geq k-1$.

Theorem 4.125 ([159]) (1) *In any rainbow* P_5*-free coloring of* $K_{n,n}$ $(n \geq 3)$ *using at least five colors, there is a monochromatic* $K_{1,2}$;

(2) *For* $t \geq 3$ *and* $k \geq t+2$, *there is a monochromatic* $K_{1,t}$ *in any rainbow* P_5*-free coloring (using all* k *colors) of* $K_{n,n}$.

Theorem 4.126 ([159]) (1) *For* $t = 2, 3, 4$, $\mathrm{bgr}_4(P_5 : K_{1,t}) = 5$.

(2) $\mathrm{bgr}_4(P_5 : K_{2,3}) = 5$; $\mathrm{bgr}_4(P_5 : K_{2,4}) = \mathrm{bgr}_5(P_5 : K_{2,4}) = 6$.

Theorem 4.127 ([159]) (1) *For* $t \geq 4$ *and* $4 \leq k \leq t+1$, $\mathrm{bgr}_k(P_5 : K_{1,t}) = t + \lfloor (t-1)/(k-2) \rfloor$.

(2) *For* $t \geq 2$ *and* $k \geq t+2$, $\mathrm{bgr}_k(P_5 : K_{2,t}) = k+1$.

Theorem 4.128 ([159]) (1) *For* $t \geq 5$ *and* $4 \leq k \leq \lfloor (t+1)/2 \rfloor + 1$, $\mathrm{bgr}_k(P_5 : K_{2,t}) = t + \lfloor (t-2)/(k-3) \rfloor$.

(2) *For* $t \geq 5$ *and* $\lfloor (t+1)/2 \rfloor + 2 \leq k \leq t+1$, $\mathrm{bgr}_k(P_5 : K_{2,t}) = t+2$.

4.6 Size Gallai-Ramsey Numbers

Define $\mathrm{gr}'_k(G : H)$ as the minimum integer N such that for all $n \geq N$, every edge-coloring of K_n using at most k colors contains either a rainbow copy of G or a monochromatic copy of H. Note that $\mathrm{gr}'_k(G : H) \leq \max\{\mathrm{gr}_i(G : H) : 1 \leq i \leq k\}$.

For two given graphs G and H, if any k-colored complete graph (using all $k \geq 4$ colors) contains either a rainbow copy of G or a monochromatic copy of H, then we call the pair (G, H) as *i-colored trivial.*

Mao [176] had the following two remarks for $\mathrm{gr}_k(G : H)$ and $\mathrm{gr}'_k(G : H)$.

Remark 1 ([176]) *If the pair* (G, H) *is* k*-color trivial, then* $\mathrm{gr}'_k(G : H) = \mathrm{gr}'_{k-1}(G : H)$. *Furthermore, if the pair* (G, H) *is* $(k-1)$*-color trivial, then* $\mathrm{gr}'_{k-1}(G : H) = \mathrm{gr}'_{k-2}(G : H)$. *Continue this process, there exists an integer* k_0 *such that the pair* (G, H) *is* k' $(k' \geq k_0+1)$*-color trivial but* (G, H) *is not* k_0*-color trivial. Then* $\mathrm{gr}'_k(G : H) = \mathrm{gr}'_{k_0}(G : H)$.

Remark 2 ([176]) *We assume that there are k colors, say $1, 2, \ldots, k$. Without loss of generality, we assume that the pair (G, H) is i_j $(k_0 < j \leq k)$-color trivial, but not i_j $(1 \leq j \leq k_0)$-color trivial. Then* $\mathrm{gr}'_k(G : H) = \max\{\mathrm{gr}_{i_j}(G : H) \,|\, 1 \leq j \leq k_0\}$.

It is natural to introduce the concept of Gallai-Ramsey number.

Definition 9 ([176]) *The k-*colored size Gallai-Ramsey number $\hat{\mathrm{gr}}_k(G, H)$ *of two graphs G and H is defined as follows:*

$$\hat{\mathrm{gr}}_k(G, H) = \min\{|E(F)| : \ F \xrightarrow{\mathrm{gr}_k} (G, H)\}.$$

Definition 10 ([176]) *Define $\hat{\mathrm{gr}}'_k(G : H)$ as the minimum integer $e(F)$ such that every k-edge-coloring of F using at most k colors contains either a rainbow copy of G or a monochromatic copy of H.*

Note that $\widehat{\mathrm{gr}}'_k(G : H) \leq \max\{\widehat{\mathrm{gr}}_i(G : H) : 1 \leq i \leq k\}$.
If $G = K_s$ and $H = K_t$, then we denote $\hat{\mathrm{gr}}_k(K_s, K_t)$ by $\hat{\mathrm{gr}}_k(s, t)$.

Proposition 10 ([176]) *Let G, H be two graphs, and let k be an integer with $k \geq 3$. If $k \geq e(G)$, then*

$$\hat{\mathrm{gr}}_k(G : H) = k.$$

Remark 3 ([176]) *From Proposition 10, $\hat{\mathrm{gr}}_k(G : H)$ is trivial for the case $k \geq e(G)$. In this paper, we assume that $k < e(G)$.*

Erdös, Faudree, Rousseau, and Schelp [60] introduced two problems of Ramsey number and size Ramsey number, and then gave nice results on them. Inspired by their idea, we will propose the following two questions similarly.

There are two preliminary questions concerning the size Gallai-Ramsey number which should be answered before posing others. For the purpose of comparing gr_k and $\hat{\mathrm{gr}}_k$, we define $\hat{\mathrm{GR}}_k(G : H)$ by

$$\hat{\mathrm{GR}}_k(G : H) = \binom{\mathrm{gr}_k(G : H)}{2}. \tag{2}$$

It is natural to ask whether there can be a large gap between $\hat{\mathrm{gr}}_k(G_s : H_t)$ and $\hat{\mathrm{GR}}_k(G_s : H_t)$ for some graphs G and H. To see the gap formally, we define the following: Let $(\{G_s\}, \{H_t\})$ be a pair of two infinite sequences of graphs. Then $(\{G_s\}, \{H_t\})$ is called a *Gallai-Ramsey o-sequence* if

$$\hat{\mathrm{gr}}_k(G_s : H_t) = o(\hat{\mathrm{GR}}_k(G_s : H_t)) \ (s \longrightarrow \infty, \ t \longrightarrow \infty). \tag{3}$$

Problem 1. (i) Do there exist graphs G, H such that $\hat{\mathrm{gr}}_k(G : H) = \hat{\mathrm{GR}}_k(G : H)$?

Problem 2. (ii) Do there exist Gallai-Ramsey o-sequences?

We can derive upper and lower bounds for $\hat{\mathrm{gr}}_k(G : H)$.

Theorem 4.129 ([176]) *Let G, H be two graphs. For $k \geq 2$ and $k < e(G)$,*

$$\frac{\delta(H)}{2} \cdot \mathrm{r}_k(H) \leq \hat{\mathrm{gr}}_k(G:H) = \hat{\mathrm{r}}_k(H) \leq \binom{\mathrm{r}_k(H)}{2}.$$

Moreover, the upper bounds is sharp.

Theorem 4.130 ([176]) *Let G, H be two graphs, and let k be an integer with $k \geq 3$. If $k < e(G)$, then*

$$k(e(H)-1)+1 \leq \hat{\mathrm{gr}}_k(G:H) \leq \binom{\mathrm{gr}_k(G:H)}{2}.$$

Moreover, the bounds are sharp.

Question (i) is answered by the following theorem, which shows that the upper bound in Theorem 4.130.

Theorem 4.131 ([176]) *For all positive integers t and a graph G, if $k < e(G)$, then*

$$\hat{\mathrm{gr}}_k(G:K_t) = \hat{\mathrm{GR}}_k(G:K_t) = \binom{\mathrm{gr}_k(G:K_t)}{2}.$$

To show the sharpness of lower bound, they considered the following graphs.

Proposition 11 ([176]) *Let G be a graph, and let k, t be two positive integers with $3 \leq k < e(G)$ and $t \geq 4$. Then*

$$\hat{\mathrm{gr}}_k(G:\ K_{1,t}) = \hat{\mathrm{gr}}_k(G:t\ K_2) = k(t-1)+1.$$

The following corollary is immediate.

Corollary 4.132 ([176]) (1) *If $s > k$ and t, k are even, then $\hat{\mathrm{gr}}_k(K_{1,s}:K_{1,t}) = \mathrm{gr}_k(K_{1,s}:K_{1,t})$.*

(2) *If $s > k$, then $(K_{1,s}, K_{1,t})$ is a Gallai-Ramsey o-sequence.*

For a fixed probability of receiving colors for each edge, they derived a lower bound of $\hat{\mathrm{gr}}_k(s,t)$.

Theorem 4.133 ([176]) *Let k, s, t be three positive integers with $2 \leq k \leq \binom{s}{2}-1$ and $s, t \geq 3$. Then*

$$\hat{\mathrm{gr}}_k(s,t) = \hat{\mathrm{r}}_k(t) \geq \frac{1}{8}\left[\frac{2t}{e} \cdot k^{\left(\frac{t-1}{2}-\frac{1}{t}\right)}+1\right]^2 - 1.$$

For a flexible probability of receiving colors for each edge, we can derive the following lower bound of $\hat{\mathrm{gr}}_k(s,t)$ by Lovász Local Lemma.

Theorem 4.134 ([176]) *Let k, s, t be three positive integers with $r, s \geq 6$ and $2 \leq k \leq \binom{s}{2}-1$. Then*

$$\hat{\mathrm{gr}}_k(s,t) = \hat{\mathrm{r}}_k(t) > \frac{1}{2}\left(\frac{t-1}{\beta c_2 \ln(t-1)}\right)^{\beta},$$

where β is a constant.

For a fixed probability of receiving colors for each edge, they derived a lower bound of $\hat{\mathrm{gr}}_k(G,H)$ for two graphs G and H.

Theorem 4.135 ([176]) *Let G, H be two graphs with s, t vertices and m_s, m_t edges, respectively. For $2 \leq k < m_s$, we have*

$$\hat{\mathrm{gr}}_k(G,H) = \hat{\mathrm{r}}_k(H) \geq \frac{\ell}{2e}\left(Yk^{1-m_t}\right)^{(-1)/\ell},$$

where

$$Y = \binom{\binom{t}{2}}{m_t} - \sum_{i=1}^{t-y^*}\binom{t}{i}\binom{\binom{t-i}{2}}{m_t}.$$

For general graphs, they gave lower bound for Gallai-Ramsey number.

Theorem 4.136 ([176]) *Let G be a graph of order $s \geq 4$, and H be a complete graph of order $t \geq 4$ and size m_s, m_t, respectively. Let c_1, c_2, c_3 be three numbers with $c_3+c_2-c_1c_2^2 < 0$. For $2 \leq k < m_s$,*

$$\hat{\mathrm{gr}}_k(G,H) = \hat{\mathrm{r}}_k(H) > \frac{1}{2}\left(\frac{2(m_t-1)^{1/2}}{c_2\beta \cdot \ln(m_t-1)}\right)^{\beta},$$

where

$$Y = \binom{\binom{t}{2}}{m_t} - \sum_{i=1}^{t-y^*}\binom{t}{i}\binom{\binom{t-i}{2}}{m_t}.$$

4.7 Gallai Colorings and Other Properties

In [51], Chen and Li considered using a color degree condition in Gallai colored complete graphs to find long rainbow paths. Recall that δ^c denotes the minimum, over all vertices $v \in V(G)$, number of colors on the edges incident to v.

Theorem 4.137 ([51]) *Any Gallai colored complete graph G has a rainbow path of length at least $\delta^c(G)$.*

For general graphs, Chen and Li also proved the following.

Theorem 4.138 ([51]) *Any Gallai colored graph G with $\delta^c(G) \geq k \geq 6$ has a rainbow path of length at least $\frac{3k}{4}$.*

Gallai colored non-complete graphs still have large monochromatic connected subgraphs.

Theorem 4.139 ([107]) *Every Gallai colored graph G contains a monochromatic connected subgraph of order at least $(\alpha(G)^2 + \alpha(G) + 1)^{-1}|G|$ vertices.*

In fact, Gallai colorings contain almost spanning highly connected subgraphs but more forbidden rainbow subgraphs have this property as well.

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Theorem 4.140 ([88]) *Let $\mathscr{H}$ be the set of all graphs H such that if G is a colored K_n containing no rainbow copy of H, then G contains a monochromatic k-connected subgraph of order at least $n - f(k, H)$ where f is a function not depending on n. Then $\mathscr{H} = \{K_3, P_6, P_4^+\}$ (and the connected subgraphs of these graphs) where P_4^+ is a path on 4 vertices with a pendant edge hanging off of one of the interior vertices.*

More generally, the following holds for hypergraphs.

Theorem 4.141 ([107]) *If the edges of an r-uniform hypergraph H are colored so that there is no rainbow copy of a fixed F, then there is a monochromatic connected subhypergraph of order at least $c|H|$ where c is a function only of F, r and $\alpha(H)$.*

Another general graph result is the following. This result was actually proven by showing that for any integer β, there exists an integer $h = h(\beta)$ such that if D is a multipartite digraph with no cyclic triangles and the largest independent set of vertices in different partite sets is β, then the smallest number of partite sets needed to dominate D is at most 4.

Theorem 4.142 ([109]) *The vertices of a Gallai colored graph G can be covered by the vertices of at most k monochromatic components where k depends only on $\alpha(G)$.*

In Section 2.10, we introduced the Erdős-Gyárfás function $f(n, p, q)$. In [158], Li, Broersma and Wang investigated the Erdős-Gyárfás function within the framework of Gallai-colorings. A Gallai-coloring of the complete graph K_n is said to be a *Gallai-(p, q)-coloring* if every K_p receives at least q distinct colors. They defined $g(n, p, q)$ to be the minimum number of colors that are needed for K_n to have a Gallai-(p, q)-coloring. Clearly, we have $f(n, p, q) \leq g(n, p, q)$ if both functions are defined for these values of n, p and q.

For studying $g(n, p, q)$, Li, Broersma and Wang introduced the following function. For $1 \leq q \leq \binom{p}{2}$, let $g_q^k(p)$ be the smallest positive integer n such that every Gallai-k-coloring of K_n contains a copy of K_p receiving at most q distinct colors. Restated, $g_q^k(p) - 1$ is the largest positive integer n such that there is a Gallai-k-coloring of K_n in which every K_p receives at least $q + 1$ distinct colors, i.e., such that $g(n, p, q + 1) \leq k$. Note that $g_1^k(p)$ is in fact the Gallai-Ramsey number $\mathrm{gr}'_k(K_3 : K_p)$.

Li, Broersma and Wang [158] pointed out that $g_q^k(p)$ is nontrivial only for $1 \leq q \leq p - 2$ (equivalently, $g(n, p, q)$ is nontrivial only for $2 \leq q \leq p - 1$). When $q \geq p - 1$, they deduced $g_q^k(p)$ using an anti-Ramsey result.

Corollary 4.143 ([158]) *For integers $k \geq 1$, $p \geq 3$ and $q \geq p - 1$, there is no Gallai-k-coloring of K_n in which every K_p receives at least $q + 1$ distinct colors. Thus $g_q^k(p) = p$ for $q \geq p - 1$.*

Moreover, if $k < q$, then it is obvious that $g_q^k(p) = p$. In the sequel, we will always assume that $k \geq q$ and $1 \leq q \leq p - 2$ when we consider $g_q^k(p)$. Note that we have the following inequalities:

$$g_q^k(p) \leq g_q^{k+1}(p), \quad g_{q+1}^k(p) \leq g_q^k(p) \; \text{ and } \; g_{q+1}^{k+1}(p) \leq g_q^k(p).$$

They obtained the following general result.

Theorem 4.144 ([158]) *For integers p, q, k with $p \geq 3$, $1 \leq q \leq p-2$ and $k \geq q$, we have $g_q^k(p) \leq 2^{\frac{2k(p-2)}{q}+1}$.*

When $q = p-2$, they proved the following result.

Theorem 4.145 ([158]) *For integers $p \geq 4$ and $k \geq p-2$, we have $g_{p-2}^k(p) = k+2$.*

The above result is equivalent to $g(n, p, p-1) = n-1$, where $n \geq p \geq 4$. Using Theorem 4.145, they showed that $g_q^k(p)$ is at least quadratic in k for $q = \lfloor\sqrt{p-1}\rfloor - 1$.

Theorem 4.146 ([158]) *For integers $p \geq 17$ and $k \geq \lfloor\sqrt{p-1}\rfloor - 1$, we have $g^k_{\lfloor\sqrt{p-1}\rfloor-1}(p) \geq k^2 + 2k + 2$.*

Note that Theorem 4.146 implies that $g(n, p, \lfloor\sqrt{p-1}\rfloor) \leq \lceil\sqrt{n}\rceil - 1$ for $p \geq 17$ and $n \geq \left(\lfloor\sqrt{p-1}\rfloor + 1\right)^2$. When $q = p-3$, Li, Broersma, and Wang [158] proved the following result, which is equivalent to $g(n, p, p-2) = n-2$ for $n \geq p \geq 8$.

Theorem 4.147 ([158]) (1) *For integers $p \geq 8$ and $k \geq p-3$, we have $g_{p-3}^k(p) = k+3$.*
(2) *For integers $k \geq 2$, we have $g_2^k(5) = 2^k + 1$.*

Theorem 4.148 ([158]) *For integers $p \geq 5$ and $k \geq \lfloor\log_2(p-1)\rfloor$, we have $g^k_{\lfloor\log_2(p-1)\rfloor}(p) \geq 2^k + 1$.*

Moreover, they obtained the following result.

Theorem 4.149 ([158]) *For integers c, p and k with $c \geq 1$, $p \geq 2(8+c)^{c+1} - 1$ and $k = p-c$, we have $g_{k-1}^k(p) = p+1$.*

5 Other Generalizations

5.1 Sub-Ramsey Theory

Definition 11 *Given a graph G and a positive integer k, the sub-Ramsey number $sr(G, k)$ is said to be the minimum number n such that if the edges of K_n are colored with no color appearing more than k times, then the colored graph contains a rainbow G.*

When the edges of K_n are colored with no color appearing more than k times, define a new edge coloring with at most k colors with each new color class containing at most one edge from each original color class. In the new edge-coloring, if there exists a monochromatic G, then it corresponds to a rainbow G in the original edge-coloring. Therefore we know that $sr(G, k) \leq r(G_1, G_2, \cdots G_k)$, where $G_i \approx G$ for all $1 \leq i \leq k$, so in general, $sr(G, k)$ is finite for any graph G and any positive integer k.

Galvin [92] gave a result on the sub-Ramsey problem, that is, $sr(K_3, k) = k+2$. For the complete graph, Hell and Montellano-Ballesteros [117] showed that $cn^{3/2} \leq sr(K_n, k) \leq$

$(2n-3)(n-2)(k-1)+3$ for some constant c, which improves upon a result due to Alspach, Gerson, Hahn and Hell [9].

The sub-Ramsey number of a cycle or a path have also been considered. Hahn and Thomassen [112] conjecture that there exists a linear function f such that $sr(C_n, k) = n$ for $k \leq f(n)$. They showed that k could grow as fast as $n^{1/3}$, and this was improved by Frieze and Reed [82] to $\frac{n}{\ln n}$ for sufficiently large n. Recently Albert, Frieze and Reed [2] settled the conjecture by Hahn and Thomassen; they showed that if n is sufficiently large and $k \leq cn$ for $c < \frac{1}{32}$, then $sr(C_n, k) = n$.

For graphs other than the complete graph, cycle and path, Hahn [111] and Fraisse, Hahn and Sotteau [81] studied the sub-Ramsey number of a star. On the other hand, sub-Ramsey number for arithmetic progressions are also studied [5, 22].

5.2 Monochromatic Degree

Let k and d be positive integers and n be a sufficiently large integer. An edge coloring of a graph G is called a (k, d)-coloring if it uses k colors and each vertex has degree at least d in each color. Given a graph F and $k \geq E(F)$, for any $n > k$, let $d(n, F, k)$ denote the minimum integer d such that every (k, d)-coloring of K_n contains a rainbow copy of F. If there is no such d, we say $d(n, F, k) = \infty$.

This topic was first studied in [72], this topic was recently revisited by Tuza in [215] with the following problems.

Problem 1 ([215]) *Given a graph F and $k \geq E(F)$, describe the behavior of $d(n, f, k)$ as a function of n.*

Problem 2 ([215]) *Does every $(k, \lfloor (n-1)/k \rfloor)$-coloring of K_n contain all graphs with fewer than k edges as rainbow subgraphs?*

Any counterexample to an affirmative answer to this claim must satisfy some necessary conditions [72].

Problem 3 ([215]) *Characterize those graphs F with the property that for $k = |E(F)|$, a rainbow copy of F occurs in every (k, d)-coloring of K_n for all sufficiently large n where*

(i) $d = \lfloor (n-1)/k \rfloor$;

(ii) $d \leq (1-c)n/k$ for some constant $c > 0$;

(iii) $d = 1$.

Some partial results were proven in [72]. In particular, there is the following case.

Problem 4 ([215]) *Is every tree in category (iii) of Problem 3?*

5.3 Multicolor Turán Problems

Definition 12 *For a graph H, the Turán number $ex(n, H)$ of H is the maximum number of edges in a graph G of order n not containing H. Such G is called an extremal H-free graph.*

Definition 13 *For an integer $k \geq 1$, let $ex_k(n, H)$ be the maximum number of edges in a multigraph G of order n such that G is an edge-colored graph using k colors and G does not contain a rainbow H.*

A simple k-coloring of a multigraph G is a decomposition of the edge multiset as the sum of k simple graphs, called "colors."

Keevash, Saks, Sudakov, and Verstraete obtained the following results.

Theorem 5.1 ([142]) *Let H be a graph, let k, n be integers with $k \geq \binom{n}{2} - ex(n, H) + |E(H)|$ and let G be a simply k-colored multigraph containing no rainbow H. Then $|E(G)| \leq k \cdot ex(n, H)$ and the equality holds only if all colors of G are identical extremal H-free graphs.*

Theorem 5.2 ([142]) *Let $r \geq 2, k \geq \binom{r}{2}, n > 10^4 r^{34}$. Then $ex_k(n, K_r) = k \cdot ex(n, K_r)$ for $k \geq (r^2 - 1)/2$ and $ex_k(n, K_r) = (\binom{r}{2} - 1) \cdot \binom{n}{2}$ for $\binom{r}{2} \leq k < (r^2 - 1)/2$.*

Theorem 5.3 ([142]) *There exists two constants $c < C$ such that, for infinitely many values of n, $ex_k(n, C_4) = k \cdot ex(n, C_4)$ for $k > C\sqrt{n}$ and $ex_k(n, C_4) = 3 \cdot \binom{n}{2}$ for $4 \leq k < c\sqrt{n}$. Moreover, for $4 \leq k < c\sqrt{n}$, an extremal simply k-colored multigraph containing no rainbow C_4 has exactly 3 non-empty colors, all of which are K_n, and for $k > C\sqrt{n}$, all the colors of an extremal simply k-colored multigraph are identical extremal C_4-free graphs.*

5.4 Others

Let $f(n)$ be the minimum number such that there is a proper edge coloring of K_n with $f(n)$ colors with no path or cycle of four edge using one or two colors, Axenovich [10] proved that $\frac{1+\sqrt{5}}{2}n - 3 \leq f(n) \leq 2n^{1+c/\sqrt{\log n}}$ for a positive constant c.

Based on Voloshin's definition in [216], several groups [23, 48, 124, 146, 211, 212] have worked on coloring the vertices of hypergraphs to avoid both monochromatic and rainbow hyper-edges.

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