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## Metric Theory of Gravity with Torsion in an Extra Dimension

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**Metric theory of gravity with torsion in an extra dimension**Karthik H. Shankar,<sup>1</sup> Anand Balaraman,<sup>2</sup> and Kameshwar C. Wali<sup>3</sup><sup>1</sup>*Center for Memory and Brain, Boston University, Boston, Massachusetts 02215, USA*<sup>2</sup>*Department of Physics, Georgia Southern University, Statesboro, Georgia 30458, USA*<sup>3</sup>*Department of Physics, Syracuse University, Syracuse, New York 13244, USA*

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We consider a theory of gravity with a hidden extra dimension and metric-dependent torsion. A set of physically motivated constraints are imposed on the geometry so that the torsion stays confined to the extra dimension and the extra dimension stays hidden at the level of four-dimensional geodesic motion. At the kinematic level, the theory maps onto general relativity, but the dynamical field equations that follow from the action principle deviate markedly from the standard Einstein equations. We study static spherically symmetric vacuum solutions and homogeneous-isotropic cosmological solutions that emerge from the field equations. In both cases, we find solutions of significant physical interest. Most notably, we find positive mass solutions with naked singularity that match the well-known Schwarzschild solution at large distances but lack an event horizon. In the cosmological context, we find an oscillatory scenario, in contrast to the inevitable singular big bang of the standard cosmology.

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**I. INTRODUCTION**

Einstein viewed space-time as a pseudo-Riemannian differentiable manifold in order to generalize the special-relativistic flat space-time to include gravity. This was primarily motivated by the fact that the local flatness of the manifold structure naturally implemented his principle of equivalence. The generalization came along with the revolutionary idea that the trajectory of any freely moving test body is simply a geodesic in the curved manifold and that gravity is not a Newtonian instantaneous action force but an effect of the curvature of the space-time manifold.

The basic constituents of the manifold structure are the metric, which defines the distance between any two points of the manifold, and the connection, which defines the covariant derivative and the curvature of the manifold. Any theory of gravity should couple the dynamics of these quantities to the dynamics of the matter moving in the space-time manifold. Among the existing theories, the ensuing field equations of general relativity (GR) are perhaps the simplest. Since in GR, torsion, the antisymmetric combination of connection coefficients, is identically zero, and since GR has withstood numerous precise experimental tests [1,2], introduction of torsion has seemed superfluous except in the presence of matter with intrinsic spin as in Einstein-Cartan formulations [3–7].

However, for two major reasons, alternate theories of gravity that reduce to GR in the weak field limit are seriously pursued. The first reason is that GR leads to inevitable singularities—black holes (death of a massive star) and big bang (birth of the universe). Though it is conventionally assumed that quantization would eliminate these singularities, GR is not readily amenable to quantization. The second reason is that the standard model of cosmology based on GR requires most of the universe to be

composed of unknown dark energy in order to account for various cosmological observations [8]. A common strategy to construct modified theories of gravity is to make the Lagrangian density a nontrivial function of the Ricci scalar [9,10] and use the action principle to derive the modified field equations. Another common strategy is to introduce extra dimensions while constraining the physical particles to a  $(3 + 1)$ -dimensional hypersurface as in the brane-world theories [11–13]. In this paper we explore a different approach by introducing metric-dependent torsion in Kaluza-Klein-type theories [14,15] with one extra dimension.

In our approach, we consider a five-dimensional (5D) manifold foliated by a family of 4D hypersurfaces, whose geometries are virtually indistinguishable from that of the 4D space-time of GR. The axis of foliation is special in the sense that there could exist nonvanishing torsion components along that dimension. We impose constraints on the connection so that any motion in the fifth dimension does not affect observations based on the geodesic motions along the 4D hypersurfaces, thus keeping the fifth dimension essentially hidden. The imposed constraints determine uniquely all the nonvanishing torsion components in terms of the 5D metric fields, making this a purely metric theory of gravity. Besides uniquely determining the torsion in the 5D geometry, the imposed constraints lead to interesting equivalence between the 5D geometry with torsion and the torsion-free 4D geometry of GR. In particular, it turns out that the components of the connection and the Ricci tensor along the 4D hypersurfaces exactly match what would arise from GR on a 4D space-time. Consequently, any test of this theory based on geodesic motions will yield the same results as GR.

Though, by construction, the extra dimension is hidden at the level of geodesic motion, its effect is clearly reflected in the field equations. The field equations are obtained by imposing the constraints on the action and varying it with

respect to the metric. This leads to global solutions that are qualitatively distinct from those obtained from GR. Most notably, we find positive mass naked singularity solutions that match the Schwarzschild solution at large distances but lack an event horizon. In the cosmological context, we find oscillatory solutions, in contrast to the inevitable singular big bang of the standard cosmology.

We begin in Sec. II with a review of the general framework of the 5D geometry. Section III deals with the specification of the constraints and the determination of the torsion and connection in terms of the metric. Section IV is devoted to the derivation of modified Einstein equations from the standard action principle using the Ricci scalar as the Lagrangian density. In Sec. V, we apply the modified Einstein equations to the homogeneous and isotropic cosmology and identify numerical solutions pointing to accelerating and oscillatory solutions to the universe. In Sec. VI, we discuss static spherically symmetric vacuum solutions and demonstrate the existence of positive mass naked singularity solutions. The final section is devoted to a summary and discussion of the results.

## II. GENERAL FRAMEWORK OF 5D GEOMETRY

We denote the coordinates of the 5D manifold by the Latin indices,  $i, j, k, \dots$  that take values 0, 1, 2, 3, and 5, and the coordinates along the 4D hypersurfaces by the Greek indices,  $\mu, \nu, \lambda, \dots$  that take values 0, 1, 2, and 3. Figure 1 is a schematic representation of the 5D geometry. With  $x^5$  denoting the axis of foliation, the metric of the foliated 5D geometry has the form

$$\mathbf{g}_{ij} = \begin{bmatrix} g_{\mu\nu} + \epsilon A_\mu A_\nu \Phi^2 & \epsilon A_\mu \Phi^2 \\ \epsilon A_\nu \Phi^2 & \epsilon \Phi^2 \end{bmatrix}, \quad (1)$$

$$\mathbf{g}^{ij} = \begin{bmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & A_\lambda A^\lambda + \epsilon \Phi^{-2} \end{bmatrix}. \quad (2)$$

$$\begin{aligned} \mathbf{g}_{\mu\nu} &= g_{\mu\nu} + \epsilon A_\mu A_\nu \Phi^2, & \mathbf{g}_{\mu 5} &= \epsilon A_\mu \Phi^2, & \mathbf{g}_{55} &= \epsilon \Phi^2, \\ \mathbf{g}^{\mu\nu} &= g^{\mu\nu}, & \mathbf{g}^{\mu 5} &= -A^\mu, & \mathbf{g}^{55} &= A_\lambda A^\lambda + \epsilon \Phi^{-2}. \end{aligned} \quad (3)$$

Here  $A_\mu$  is a 4D vector whose indices are raised and lowered with respect to the 4D metrics  $g^{\mu\nu}$  and  $g_{\mu\nu}$ . The fifth dimension is spacelike if  $\epsilon = +1$ , and it is timelike if  $\epsilon = -1$ . Note that the 5D metric is denoted by  $\mathbf{g}$ , and the 4D metric is  $g$ .

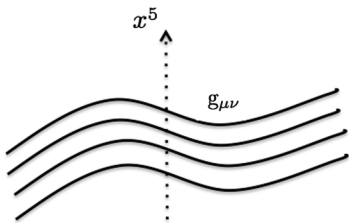


FIG. 1. Schematic representation of the 5D geometry.

Let us denote the connection in the 5D geometry by  $\tilde{\Gamma}$ . and its antisymmetric part, the torsion, by  $T$ .

$$T^i_{\cdot jk} = \tilde{\Gamma}^i_{\cdot jk} - \tilde{\Gamma}^i_{\cdot kj}. \quad (4)$$

Denoting the covariant derivative induced by the connection by  $\tilde{\nabla}$ , the metricity condition is expressed as  $\tilde{\nabla}_k \mathbf{g}_{ij} = 0$ . With the metricity condition, the connection  $\tilde{\Gamma}$  can be expressed as a sum of the Levi-Civita connection  $\hat{\Gamma}$  and the contorsion  $K$ ,

$$\tilde{\Gamma}^i_{\cdot jk} = \hat{\Gamma}^i_{\cdot jk} + K^i_{jk}, \quad (5)$$

where the Levi-Civita connection is expressed purely in terms of the metric

$$\hat{\Gamma}^i_{\cdot jk} = \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} = \frac{1}{2} \mathbf{g}^{im} [\partial_j \mathbf{g}_{km} + \partial_k \mathbf{g}_{jm} - \partial_m \mathbf{g}_{jk}], \quad (6)$$

and the contorsion is expressed in terms of the torsion [16]

$$K^i_{jk} = \frac{1}{2} [T^i_{\cdot jk} + T^i_{\cdot kj} + T^i_{\cdot k j}]. \quad (7)$$

In the absence of torsion, the connection is simply the Levi-Civita part. In order to compare the dynamics of this geometry to GR, we consider a reference space-time in four dimensions with the metric  $g_{\mu\nu}$  and torsion-free 4D Levi-Civita connection  $\Gamma^{\lambda}_{\mu\nu}$ ,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} [\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}]. \quad (8)$$

This connection is different from the 4D components of the 5D Levi-Civita connection that contains additional terms  $\tilde{\Gamma}^{\lambda}_{\mu\nu}$  [see Eq. (A7)] that depend on the extra-dimensional metric fields  $A_\mu$  and  $\Phi$ . Hence the 4D components of the 5D Levi-Civita connection can be written as

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tilde{\Gamma}^{\lambda}_{\mu\nu}. \quad (9)$$

In the presence of torsion, with the inclusion of contorsion, the 4D components of the 5D connection take the form

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tilde{\Gamma}^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu}. \quad (10)$$

We note that the additional terms  $(\tilde{\Gamma}^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu})$  do not generally vanish. However, in the next section we impose constraints on the connection and find that these terms do vanish.

## III. CONSTRAINTS ON THE CONNECTION

With minimal modifications to standard GR in mind, we first assume that the 4D components of the connection  $\tilde{\Gamma}^{\lambda}_{\mu\nu}$  are symmetric, that is, (i)  $T^{\lambda}_{\mu\nu} = 0$ . Next, we require that geodesic motion and its observable effects in 4D are not affected by any motion in the fifth dimension. This requirement essentially ensures that the fifth dimension stays hidden at the level of 4D geodesics. For this purpose, considering the 4D components of the geodesic equations in the 5D geometry, namely,

$$\ddot{x}^\lambda + \tilde{\Gamma}^\lambda_{\cdot\mu\nu} \dot{x}^\mu \dot{x}^\nu + (\tilde{\Gamma}^\lambda_{\cdot\mu 5} + \tilde{\Gamma}^\lambda_{\cdot 5\mu}) \dot{x}^\mu \dot{x}^5 + \tilde{\Gamma}^\lambda_{\cdot 55} (\dot{x}^5)^2 = 0,$$

we are led to the second constraint, (ii)  $\tilde{\Gamma}^\lambda_{\cdot i5} = \tilde{\Gamma}^\lambda_{\cdot 5i} = 0$ . An alternative formulation of these constraints in terms of vielbeins is worked out in [17]. These constraints are clearly not tensorial in nature because the fifth dimension is singled out. It turns out, however, that they are sufficient to determine uniquely all the nonvanishing torsion components in terms of the metric (see Appendix A for details).

$$\begin{aligned} T^\mu_{\cdot ij} &= 0, \\ T^5_{\cdot \mu\nu} &= 2\partial_{[\mu} A_{\nu]} + 2J_{[\mu} A_{\nu]}, \\ T^5_{\cdot \mu 5} &= J_\mu - \partial_5 A_\mu - A_\mu J_5, \end{aligned} \quad (11)$$

where  $J_i \equiv \Phi^{-1} \partial\Phi / \partial x^i$ .

Using the above results for torsion and Eqs. (5)–(7), we find the connection coefficients

$$\begin{aligned} \tilde{\Gamma}^\lambda_{\cdot 55} &= \tilde{\Gamma}^\lambda_{\cdot \nu 5} = \tilde{\Gamma}^\lambda_{\cdot 5\nu} = 0, & \tilde{\Gamma}^5_{\cdot \mu\nu} &= \nabla_\mu A_\nu + J_\mu A_\nu, \\ \tilde{\Gamma}^5_{\cdot 5\mu} &= \partial_5 A_\mu + J_5 A_\mu, & \tilde{\Gamma}^5_{\cdot \mu 5} &= J_\mu, & \tilde{\Gamma}^5_{\cdot 55} &= J_5, \\ \tilde{\Gamma}^\lambda_{\cdot \mu\nu} &= \Gamma^\lambda_{\cdot \mu\nu}. \end{aligned} \quad (12)$$

Here  $\nabla_\mu$  is the covariant derivative operator in the torsion-free 4D geometry with metric  $g_{\mu\nu}$ . This connection has a very special property: its 4D components are exactly the 4D Levi-Civita connection. That is,  $\tilde{\Gamma}^\lambda_{\cdot \mu\nu}$  and  $K^\lambda_{\cdot \mu\nu}$  in Eq. (10) exactly cancel each other.

In addition to determining the torsion and the connection in terms of the metric fields, the constraints also imply that the 4D metrics on all the hypersurfaces are identical. As a consequence, the 4D components of the connection do not depend on  $x^5$  either,

$$\frac{\partial g_{\mu\nu}}{\partial x^5} = 0 \Rightarrow \frac{\partial \tilde{\Gamma}^\lambda_{\cdot \mu\nu}}{\partial x^5} = 0. \quad (13)$$

This should be contrasted with the Kaluza-Klein-type theories where it is *a priori* assumed that  $g_{\mu\nu}$ ,  $A_\mu$ , and  $\Phi$  are independent of  $x^5$ , known as the cylindrical condition. In our framework, though  $g_{\mu\nu}$  is required to be independent of  $x^5$ ,  $A_\mu$  and  $\Phi$  can, in principle, depend on  $x^5$ .

Substituting the connection [Eq. (12)] in the Ricci tensor defined by

$$\tilde{R}_{ik} = \partial_k \tilde{\Gamma}^j_{\cdot ji} - \partial_j \tilde{\Gamma}^j_{\cdot ki} + \tilde{\Gamma}^j_{\cdot km} \tilde{\Gamma}^m_{\cdot ji} - \tilde{\Gamma}^j_{\cdot jm} \tilde{\Gamma}^m_{\cdot ki}, \quad (14)$$

we find

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}, \quad \tilde{R}_{\mu 5} = \tilde{R}_{5\mu} = \tilde{R}_{55} = 0. \quad (15)$$

Here  $R_{\mu\nu}$  represents the Ricci tensor constructed from the torsion-free 4D Levi-Civita connection. Hence the 4D components of the Ricci tensor exactly match the Ricci tensor in GR with the metric  $g_{\mu\nu}$ . It also follows that the 5D Ricci scalar is exactly the same as the Ricci scalar in the torsion-free 4D space-time, that is,  $\tilde{R} = R$ .

An important point to emphasize is that, at the level of geometry, this framework is virtually indistinguishable from the torsion-free 4D space-time of GR. Any observable geodesic motion or geodesic deviations between particles would match what we expect based on GR. However, this is true only to the extent that the metric  $g_{\mu\nu}$  is identical to the solution of the Einstein equations in GR. In Sec. VI, we will see that this is indeed true in the weak field limit for spherically symmetric vacuum solutions that are relevant for experimental observations within the solar system.

#### IV. ACTION PRINCIPLE AND MODIFIED EINSTEIN EQUATIONS

We start with the standard Einstein-Hilbert action with the Ricci scalar as the Lagrangian density,

$$S = \int \tilde{R} \sqrt{-\mathbf{g}} d^5x. \quad (16)$$

In varying the action, we note that the Ricci scalar and the connection coefficients described in the previous section are functions of the metric components alone.

$$\begin{aligned} \delta S &= \int [\tilde{R} \delta \sqrt{-\mathbf{g}} + \tilde{R}_{ik} \delta \mathbf{g}^{ik} \sqrt{-\mathbf{g}}] d^5x \\ &+ \int \delta \tilde{R}_{ik} \mathbf{g}^{ik} \sqrt{-\mathbf{g}} d^5x. \end{aligned} \quad (17)$$

The first term gives rise to the usual Einstein tensor,

$$\tilde{G}_{ik} = \tilde{R}_{ik} - (1/2) \mathbf{g}_{ik} \tilde{R}.$$

In the absence of torsion, the second term becomes a boundary integral which vanishes when the variation is fixed at the boundary and hence will not contribute to the equations of motion. But in the presence of torsion, the second term gives a nonzero contribution.

From Eq. (14), we find the variations of the Ricci tensor to be

$$\begin{aligned} \delta \tilde{R}_{ik} &= \partial_k \delta \tilde{\Gamma}^j_{\cdot ji} - \partial_j \delta \tilde{\Gamma}^j_{\cdot ki} + \tilde{\Gamma}^j_{\cdot km} \delta \tilde{\Gamma}^m_{\cdot ji} + \tilde{\Gamma}^m_{\cdot ji} \delta \tilde{\Gamma}^j_{\cdot km} \\ &- \tilde{\Gamma}^j_{\cdot jm} \delta \tilde{\Gamma}^m_{\cdot ki} - \tilde{\Gamma}^m_{\cdot ki} \delta \tilde{\Gamma}^j_{\cdot jm} \\ &= [\tilde{\nabla}_k \delta \tilde{\Gamma}^j_{\cdot ji} - \tilde{\nabla}_j \delta \tilde{\Gamma}^j_{\cdot ki}] + T^m_{\cdot kj} \delta \tilde{\Gamma}^j_{\cdot mi}. \end{aligned} \quad (18)$$

Then, the second term in the right-hand side (r.h.s.) of Eq. (17) takes the form

$$\begin{aligned} \int \delta \tilde{R}_{ik} \mathbf{g}^{ik} \sqrt{-\mathbf{g}} d^5x &= \int [\tilde{\nabla}_k (\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{\cdot ji}) \\ &- \tilde{\nabla}_j (\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{\cdot ki})] \sqrt{-\mathbf{g}} d^5x \\ &+ \int \mathbf{g}^{ik} T^m_{\cdot kj} \delta \tilde{\Gamma}^j_{\cdot mi} \sqrt{-\mathbf{g}} d^5x. \end{aligned} \quad (19)$$

In deriving the above equation, we have used the metricity condition, namely,  $\tilde{\nabla}_j \mathbf{g}^{ik} = 0$ . Substituting for the covariant derivative in the first term of the r.h.s. of Eq. (19), we obtain

$$\begin{aligned}
& \int [\tilde{\nabla}_k(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji}) - \tilde{\nabla}_j(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki})] \sqrt{-\mathbf{g}} d^5x \\
&= \int [\partial_k(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji} \sqrt{-\mathbf{g}}) - \partial_j(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki} \sqrt{-\mathbf{g}})] d^5x \\
&+ \int \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji} \left[ \tilde{\Gamma}^m_{mk} - \frac{\partial_k \sqrt{-\mathbf{g}}}{\sqrt{-\mathbf{g}}} \right] \sqrt{-\mathbf{g}} d^5x \\
&- \int \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki} \left[ \tilde{\Gamma}^m_{mj} - \frac{\partial_j \sqrt{-\mathbf{g}}}{\sqrt{-\mathbf{g}}} \right] \sqrt{-\mathbf{g}} d^5x. \quad (20)
\end{aligned}$$

The first term on the r.h.s. of the above equation is a boundary term, an integral of a total divergence. This will vanish when the variation is fixed at the boundary, and hence can be ignored. The second and third terms on the r.h.s. can be simplified by noting  $\tilde{\Gamma}^m_{mk} = \hat{\Gamma}^m_{mk} + T^m_{mk}$  and  $\hat{\Gamma}^m_{mk} = (\partial_k \sqrt{-\mathbf{g}}) / \sqrt{-\mathbf{g}}$ , leading to

$$\begin{aligned}
& \int [\tilde{\nabla}_k(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji}) - \tilde{\nabla}_j(\mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki})] \sqrt{-\mathbf{g}} d^5x \\
&= - \int T^m_{km} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji} \sqrt{-\mathbf{g}} d^5x \\
&+ \int T^m_{jm} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki} \sqrt{-\mathbf{g}} d^5x, \quad (21)
\end{aligned}$$

and Eq. (19) becomes

$$\begin{aligned}
\int \delta \tilde{R}_{ik} \mathbf{g}^{ik} \sqrt{-\mathbf{g}} d^5x &= \int [-T^m_{km} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji} + T^m_{jm} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki} \\
&+ T^m_{kj} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{mi}] \sqrt{-\mathbf{g}} d^5x. \quad (22)
\end{aligned}$$

The second term on the r.h.s. of Eq. (17) is thus a function of torsion and variations in the connection given by Eq. (22). Clearly, this term vanishes if we assume that torsion is zero, and the action principle would yield the standard Einstein equations. Alternatively, if we treat the variation in the connection to be composed of independent variations in the metric and torsion, we obtain the Einstein-Cartan equations [3], which ultimately lead to zero torsion when the matter is not coupled to the connection. However, since torsion is not an independent degree of freedom in our framework but it is a function of the metric components given by Eq. (11), we first substitute its components in terms of the metric and then carry out the variation with respect to the metric. To this end, we note

$$\begin{aligned}
T^m_{km} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ji} &= T^5_{\nu 5} [\mathbf{g}^{\mu\nu} (\delta \tilde{\Gamma}^{\alpha\mu}_{\alpha\mu} + \delta \tilde{\Gamma}^5_{5\mu}) + \mathbf{g}^{5\nu} \delta \tilde{\Gamma}^5_{55}], \\
T^m_{jm} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{ki} &= T^5_{\nu 5} \mathbf{g}^{\mu\alpha} \delta \tilde{\Gamma}^{\nu\mu\alpha}, \\
T^m_{kj} \mathbf{g}^{ik} \delta \tilde{\Gamma}^j_{mi} &= T^5_{\nu 5} [\mathbf{g}^{\mu\nu} \delta \tilde{\Gamma}^5_{5\mu} + \mathbf{g}^{5\nu} \delta \tilde{\Gamma}^5_{55}]. \quad (23)
\end{aligned}$$

Taking these terms together, Eq. (22) takes the form

$$\begin{aligned}
\int \delta \tilde{R}_{ik} \mathbf{g}^{ik} \sqrt{-\mathbf{g}} d^5x &= \int T^5_{\nu 5} [\mathbf{g}^{\mu\alpha} \delta \Gamma^{\nu\mu\alpha} \\
&- \mathbf{g}^{\mu\nu} \delta \Gamma^{\alpha\mu}_{\alpha\mu}] \sqrt{-\mathbf{g}} d^5x. \quad (24)
\end{aligned}$$

Note that the variations in the connection,  $\delta \Gamma^{\nu\mu\alpha}$  and  $\delta \Gamma^{\alpha\mu}_{\alpha\mu}$ , involve only the 4D components. Since these are the 4D Levi-Civita components that only depend on the 4D metric  $g_{\mu\nu}$ , the above equation takes the form

$$\int \delta \tilde{R}_{ik} \mathbf{g}^{ik} \sqrt{-\mathbf{g}} d^5x = \int H_{\mu\nu} \delta g^{\mu\nu} \sqrt{-\mathbf{g}} d^5x, \quad (25)$$

where (see Appendix B for details)

$$\begin{aligned}
H_{\mu\nu} &= \nabla_{(\mu} B_{\nu)} - (\nabla \cdot B) g_{\mu\nu} + J_{(\mu} B_{\nu)} - (J \cdot B) g_{\mu\nu}, \\
B_{\mu} &\equiv T^5_{\mu 5} = J_{\mu} - \partial_5 A_{\mu} - A_{\mu} J_5. \quad (26)
\end{aligned}$$

Taking together the variations in both terms in Eq. (17), we obtain the modified Einstein tensor

$$\begin{aligned}
\tilde{G}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}(g_{\mu\nu} + A_{\mu} A_{\nu} \epsilon \Phi^2) R + H_{\mu\nu} = \Sigma_{\mu\nu}, \\
\tilde{G}_{\mu 5} &= -\frac{1}{2} A_{\mu} \epsilon \Phi^2 R = \Sigma_{\mu 5}, \\
\tilde{G}_{55} &= -\frac{1}{2} \epsilon \Phi^2 R = \Sigma_{55},
\end{aligned}$$

where  $\Sigma$  is the stress-energy tensor that arises from the variations of the assumed matter fields in the Lagrangian. Our focus being on gravity, we will not discuss the origin of  $\Sigma$  further. Since the physical interpretation of the stress energy is more transparent with one covariant and one contravariant index, we express the above equations in an alternate form, by noting  $\tilde{G}_i{}^j = \mathbf{g}^{kj} \tilde{G}_{ik}$ ,

$$R_{\mu}{}^{\nu} - \frac{1}{2} R \delta_{\mu}{}^{\nu} + H_{\mu}{}^{\nu} = \Sigma_{\mu}{}^{\nu}, \quad (27)$$

$$-A^{\alpha} R_{\mu\alpha} - A^{\alpha} H_{\mu\alpha} = \Sigma_{\mu}{}^5, \quad (28)$$

$$0 = \Sigma_5{}^{\mu}, \quad -\frac{1}{2} R = \Sigma_5{}^5 \quad (29)$$

These are the modified Einstein equations in our framework. Since, by construction, the fifth dimension is hidden with respect to the observable 4D motion, the 5D components of the stress tensor,  $\Sigma_{\mu}{}^5$  and  $\Sigma_5{}^5$ , are unobservable. It is not possible to solve Eqs. (28) and (29) unless these components are theoretically known from the 5D matter Lagrangian. In the present formulation, for simplicity, we shall ignore these equations as though they simply serve to evaluate the components  $\Sigma_{\mu}{}^5$  and  $\Sigma_5{}^5$ , and treat only Eq. (27) with the observable 4D stress tensor to be relevant to physical solutions. In the absence of specified matter fields in the Lagrangian, an alternate way to interpret the modified Einstein equations is to regard  $-H_{\mu}{}^{\nu}$  as extra-dimensionally induced matter.

When  $H_{\mu}{}^{\nu} = 0$ , Eq. (27) reduces to the standard Einstein equations for the 4D metric components  $g_{\mu\nu}$ . In this case the 4D Bianchi identity necessarily implies the conservation of matter,  $\nabla_{\nu} \Sigma_{\mu}{}^{\nu} = 0$ . But, in general, when  $H_{\mu}{}^{\nu}$  is nonvanishing and dependent on the extra-dimensional metric fields  $A_{\mu}$  and  $\Phi$ , Eq. (27) by itself may not be sufficient to solve for  $g_{\mu\nu}$  along with  $A_{\mu}$  and  $\Phi$ , even after fixing the gauge. However, an important

physical simplification can be achieved by generalizing the cylindrical condition to assume that  $A_\mu$  and  $\Phi$  do not depend on  $x^5$ . With this assumption,  $B_\mu = J_\mu$  and  $H_\mu{}^\nu$  depends only on  $\Phi$  and not on  $A_\mu$ ; Eq. (27) is sufficient to solve for both  $g_{\mu\nu}$  and  $\Phi$ . The vector  $A_\mu$  can, in principle, be evaluated from Eq. (28) by setting  $\Sigma_\mu{}^\nu$  to zero, but this would be inconsequential as  $A_\mu$  is decoupled from the physically relevant equation that solves for the 4D metric  $g_{\mu\nu}$ . Hence, in the rest of the paper we will make the assumption of the cylindrical condition in order to explore solutions of physical interest to the modified Einstein equations.

Finally, when  $H_\mu{}^\nu$  is nonvanishing, we note that  $\Sigma_\mu{}^\nu$  does not necessarily have to satisfy the 4D matter conservation. However, with minimum modifications to GR and the empirical conservation laws in mind, it is reasonable to assert the conservation of  $\Sigma_\mu{}^\nu$ . Since the standard Einstein tensor satisfies the 4D Bianchi identity independently of  $H_\mu{}^\nu$ , the 4D matter conservation implies

$$\nabla_\nu \Sigma_\mu{}^\nu = 0 \Rightarrow \nabla_\nu H_\mu{}^\nu = 0. \quad (30)$$

In the remainder of the paper, we study the solutions to the modified Einstein equations [Eq. (27)] in two extremely symmetric situations, namely, the homogeneous-isotropic geometry and the static spherically symmetric geometry.

## V. HOMOGENEOUS-ISOTROPIC COSMOLOGY

The 4D metric of a homogeneous and isotropic universe has the form

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (31)$$

The values of  $k = 0, +1, -1$  correspond, respectively, to flat, closed, and hyperbolic spatial geometries. The standard Einstein tensor for this metric is given by [18]

$$\begin{aligned} G_t^t &= 3(\dot{a}/a)^2 + 3k/a^2, \\ G_r^r &= 2(\ddot{a}/a) + (\dot{a}/a)^2 + k/a^2, \\ G_\theta^\theta &= G_\phi^\phi = G_r^r, \end{aligned} \quad (32)$$

where an over-dot denotes a derivative with respect to time. Since the geometry is spatially homogeneous and isotropic, the metric fields including  $A_\mu$  and  $\Phi$  in the 5D geometry only depend on time. Hence the only nonvanishing component of  $J_\mu$  is  $J_t$ . The induced matter terms given in Eq. (26) are

$$\begin{aligned} H_t^t &= 3J_t(\dot{a}/a), \\ H_r^r &= 2J_t(\dot{a}/a) + \dot{J}_t + J_t^2, \\ H_\theta^\theta &= H_\phi^\phi = H_r^r. \end{aligned} \quad (33)$$

Before writing out the modified Einstein equations, we note that the conservation equation [Eq. (30)] now reduces to

$$\begin{aligned} \nabla_\nu H_\mu{}^\nu &= 3J_t[(\ddot{a}/a) - (\dot{a}/a)J_t] = 0 \Rightarrow \\ J_t &= 0 \quad \text{or} \quad J_t = \ddot{a}/\dot{a}. \end{aligned} \quad (34)$$

These are the only two possibilities. From the definition of  $J_\mu$ , this implies that either  $\Phi$  is a constant, which would give rise to the usual Friedmann-Robertson-Walker (FRW) cosmology, or  $\Phi = \dot{a}(t)$ . Focusing on the latter case,  $H_\mu{}^\nu$  simplifies to

$$H_t^t = 3\ddot{a}/a, \quad H_r^r = 2(\ddot{a}/a) + (\ddot{a}/\dot{a}). \quad (35)$$

Taking the stress tensor to be that of a perfect fluid, the modified Einstein equations [Eq. (27)] take the form

$$3(\dot{a}/a)^2 + 3k/a^2 + 3\ddot{a}/a = 8\pi\rho, \quad (36)$$

$$4(\ddot{a}/a) + (\dot{a}/a)^2 + (\ddot{a}/\dot{a}) + k/a^2 = -8\pi P, \quad (37)$$

where  $\rho$  and  $P$  are the density and pressure of the 4D matter. Combining the above equations, we find

$$\dot{\rho} + 3(\rho + P)\dot{a}/a = 0, \quad (38)$$

which is just a restatement of the 4D matter conservation equation. For a matter dominated universe,  $P = 0$ , and consequently Eq. (38) yields

$$\rho a^3 = \text{constant} \quad \text{or} \quad \rho = \rho_o/a^3. \quad (39)$$

In effect, Eqs. (36) and (39) are sufficient to solve for  $a(t)$ , which needs two initial conditions along with the specification of  $\rho_o$ .

$$a^2\ddot{a} + a(\dot{a})^2 + ka = 8\pi\rho_o/3. \quad (40)$$

Without loss of generality, we choose the current epoch to be at  $t = 0$ , set the current size of the universe as  $a(0) = 1$ , and set the unit of time such that  $\dot{a}(0) = 1$ . In this unit of time, the current value of the Hubble constant will be 1. Rather than specifying the value of  $\rho_o$ , we choose to specify the current value of  $\ddot{a}$ . The effective equation for  $a(t)$  then takes the form

$$a^2\ddot{a} + a(\dot{a})^2 + ka = 1 + k - q_o, \quad (41)$$

where  $q_o = -a(0)\ddot{a}(0)/\dot{a}^2(0)$ , the current value of the deceleration parameter, is the only free parameter to be specified. Figure 2 shows the behavior of  $a(t)$  for various values of  $q_o$  for spatially flat and closed topologies. For the spatially flat topology, shown in the top panels of the figure, we find that the universe does not originate from a singular big bang for all  $q_o < +0.5$ . For the spatially closed topology shown in the bottom panels, we find oscillatory solutions for all  $q_o < +1$ .

Oscillatory solutions in the spatially closed topology exhibit a scale factor that oscillates between a maximum  $a_{\max}$  and a minimum  $a_{\min}$ . The acceleration reaches a positive value in a narrow interval around  $a_{\min}$ , and then becomes negative for the rest of the cycle until it gets back near  $a_{\min}$ .

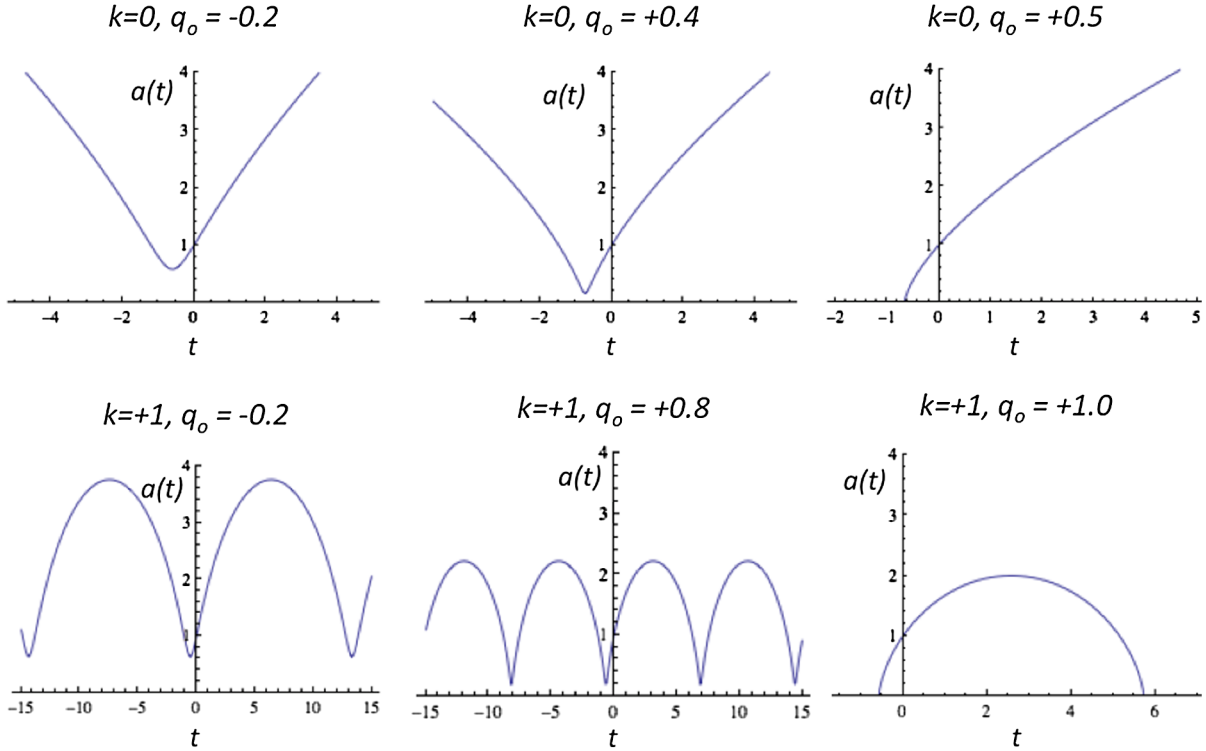


FIG. 2 (color online). Solutions for  $a(t)$  for different values of the deceleration parameter for  $k = 0$  and  $k = +1$ .

By taking the value of  $q_o$  arbitrarily close to 1, we can make  $a_{\min}$  arbitrarily close to zero. This can be seen in the bottom-left and bottom-middle panels of Fig. 2. Thus one could construct a universe that collapses and bounces back to expand when it reaches an arbitrarily small size or, equivalently, arbitrarily high energy densities. It remains to be seen if such solutions would fit the empirical redshift data.

## VI. STATIC SPHERICALLY SYMMETRIC VACUUM SOLUTIONS

The most general static spherically symmetric 4D metric has the form<sup>1</sup>

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2, \quad (42)$$

and the standard Einstein tensor for this metric is [18]

$$\begin{aligned} G_t^t &= \frac{rB' + B^2 - B}{r^2B^2}, \\ G_r^r &= \frac{AB - rA' - A}{r^2AB}, \\ G_\theta^\theta &= \frac{2A^2B' - 2ABA' - 2rABA'' + rBA'^2 + rAA'B'}{4rA^2B^2}, \\ G_\phi^\phi &= G_\theta^\theta. \end{aligned} \quad (43)$$

<sup>1</sup>The scalar functions  $A(r)$  and  $B(r)$  defined in this section should not be confused with the vectors  $A_\mu$  and  $B_\mu$  defined in Eqs. (3) and (26), respectively.

where a prime in the above equations denotes a derivative with respect to  $r$ .

The additional term  $H_\mu{}^\nu$  in the modified Einstein equations [Eq. (27)] depends only on  $\Phi$  when the cylindrical condition is imposed on all metric components, and is given by

$$H_\mu{}^\nu = \nabla_\mu J^\nu - (\nabla \cdot J)\delta_\mu^\nu + J_\mu J^\nu - (J \cdot J)\delta_\mu^\nu. \quad (44)$$

Since  $J_\mu = \Phi^{-1}\partial_\mu\Phi$ , the quantity  $\nabla_\mu J^\nu$  is intrinsically symmetric in  $\mu$  and  $\nu$ . The static spherical symmetry of the geometry implies that  $J_r$  is the only nonvanishing component, which we denote by  $J(r)$ . With this,

$$\begin{aligned} H_t^t &= \frac{rJB' - 2B(rJ' + 2J + rJ^2)}{2rB^2}, \\ H_r^r &= \frac{-J(rA' + 4A)}{2rAB}, \\ H_\theta^\theta &= \frac{rJAB' - 2rABJ' - rJBA' - 2ABJ(1 + rJ)}{2rAB^2}, \\ H_\phi^\phi &= H_\theta^\theta. \end{aligned} \quad (45)$$

In order to obtain vacuum solutions, we set  $\Sigma_\mu{}^\nu = 0$  in Eq. (27) and find the following three equations:

$$\begin{aligned} J' &= -\frac{J(1+B)}{r}, \\ A' &= -\frac{2A(1-B+2rJ)}{r(2+Jr)}, \\ B' &= \frac{2B(r^2J^2 + (1+rJ)(1-B))}{r(2+Jr)}. \end{aligned} \quad (46)$$

A close examination of the above equations reveals two basic properties of the function  $J(r)$ . First, if  $J(r)$  is a constant, it has to be identically zero. Second, if  $J(r)$  vanishes at some point, it has to vanish identically everywhere. The simplest solution to the coupled equations [Eq. (46)] is when  $J(r)$  vanishes everywhere,

$$J(r)=0, \quad A(r)=\left(1-\frac{2M}{r}\right), \quad B(r)=\left(1-\frac{2M}{r}\right)^{-1}, \quad (47)$$

which of course is the well-known Schwarzschild solution, as expected.

### A. General solution to $J(r)$

Let  $F(r) \equiv 1/rJ(r)$  when  $J(r)$  is nonvanishing. Substituting for  $J(r)$  in terms of  $F(r)$ , the coupled equations [Eq. (46)] lead to the following equation for  $F(r)$ ,

$$r[(2F^2 + F)F'' + F'^2] = F'(F + 2), \quad (48)$$

whose solution in turn determines the 4D metric functions  $A(r)$  and  $B(r)$ . The obvious solution of Eq. (48) is  $F(r) = \text{constant}$ . This leads to

$$J(r) = c/r, \quad A(r) = (r)^{-(2+4c/2+c)}, \quad B(r) = 0, \quad (49)$$

which is clearly unacceptable because  $B(r)$  is identically zero.

Assuming that  $F(r)$  is not a constant, we can obtain solutions to the second order differential equation [Eq. (48)]. In principle, the solution would have two integration constants that would be determined by the boundary conditions, one of which immediately follows from the form of the equation. It can be easily seen that if  $F(r)$  is a solution, then  $F(\lambda r)$  is also a solution for any scaling constant  $\lambda$ . We find a general solution in the implicit form

$$\lambda^2 r^2 = \gamma^2 |F/\gamma + \beta + 1|^{1+\beta} |F/\gamma + \beta - 1|^{1-\beta}, \quad (50)$$

where  $\beta$  and  $\gamma$  are defined in terms of an independent arbitrary constant  $c$ .

$$\gamma = \sqrt{1+c+c^2}, \quad \beta = \frac{1+c}{\sqrt{1+c+c^2}}. \quad (51)$$

With  $\lambda$  and  $c$  as two arbitrary constants, Eq. (50) represents the general solution to the second order differential equation [Eq. (48)]. From Eq. (50), we find the derivatives of  $F(r)$  to be

$$F' = \frac{1}{r} \frac{F^2 + 2(1+c)F + c}{F}, \quad F'' = -\frac{c}{r} \frac{F'}{F^2}, \quad (52)$$

and substituting them into Eq. (46), we obtain the metric functions  $A(r)$  and  $B(r)$  in terms of  $F(r)$ .

$$A'(r) = 2cA(r)/rF(r), \quad (53)$$

$$B(r) = 1 + 2(1+c)/F(r) + c/F^2(r). \quad (54)$$

In order to obtain asymptotically flat solutions, we shall impose the boundary conditions  $A(r \rightarrow \infty) = 1$  and  $B(r \rightarrow \infty) = 1$ . To understand the behavior of the functions  $A(r)$  and  $B(r)$  which define the observable 4D geometry, we start with the properties of  $F(r)$ .

Unfortunately Eq. (50) does not yield an explicit functional form for  $F(r)$  except for simple cases when  $c$  is either 0 or  $-1$ . Nevertheless, the relevant properties of  $F(r)$  can be inferred from analyzing this implicit function. First note that the quantity  $(1+c+c^2)$  is positive definite and  $\beta$  is finite and bounded for all values of  $c$ . Figure 3 plots the behavior of  $\beta$  to show that it asymptotically reaches  $+1$  and  $-1$  at  $c = +\infty$  and  $-\infty$ , respectively. The following observations summarize the qualitative properties of  $F(r)$ .

- (1) At  $r = 0$ ,  $F$  can take one of two possible values. If  $c < 0$  ( $\beta < 1$ ), then  $F(0)$  can be either  $\gamma(-\beta - 1)$  or  $\gamma(-\beta + 1)$ , while if  $c > 0$  ( $\beta > 1$ ), then  $F(0)$  can only be  $\gamma(-\beta - 1)$ .
- (2) In the limit  $r \rightarrow \infty$ ,  $F$  necessarily has to diverge in order to satisfy the boundary condition  $B(r \rightarrow \infty) = 1$ . Equation (50) then implies that for large  $r$ ,  $\lambda^2 r^2 = F^2$ , implying that  $F$  could be either positive or negative, such that

$$F(r \rightarrow \infty) = \lambda r \Rightarrow J(r \rightarrow \infty) = \lambda^{-1}/r^2. \quad (55)$$

The behavior of  $F(r)$  at the extremities is summarized in the following table.

	$c < 0 (\beta < 1)$	$c > 0 (\beta > 1)$
$r = 0$	$F = \begin{cases} \gamma(-\beta + 1) > 0 \\ \gamma(-\beta - 1) < 0 \end{cases}$	$F = \gamma(-\beta - 1) < 0$
$r \rightarrow \infty$	$F = \lambda r$	$F = \lambda r$

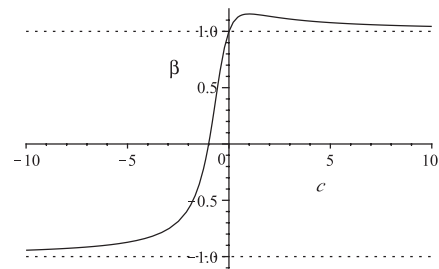


FIG. 3.  $\beta$  is plotted as a function of  $c$ .



- (3) From Eq. (52), it can be shown that  $F' = 0$  when  $F$  is either  $\gamma(-\beta + 1)$  or  $\gamma(-\beta - 1)$ , which can happen only at  $r = 0$ . Hence  $F'$  is either positive definite or negative definite, and so  $F(r)$  is a monotonic function.
- (4) Though  $F$  is monotonic and finite for any finite  $r$ , it can reach zero at  $r_o$ , given by

$$\lambda^2 r_o^2 = |c| \left| \frac{\beta + 1}{\beta - 1} \right|^\beta, \quad (56)$$

and from Eq. (52),  $F'$  diverges at  $r_o$ . Considering the physical relevance of these solutions, we shall only focus on solutions that are nonvanishing everywhere. Such solutions do indeed exist for a range of parameter values. Rewriting Eq. (50) at  $r = r_o$ ,

$$\left| \frac{F(r_o)}{\gamma(\beta + 1)} + 1 \right|^{1+\beta} \left| \frac{F(r_o)}{\gamma(\beta - 1)} + 1 \right|^{1-\beta} = 1, \quad (57)$$

we note that  $F(r_o) = 0$  is not the only solution. Numerical plots in Fig. 4 demonstrate the existence of nonvanishing  $F(r)$  solutions.

- (5) From the table above, we see that (i) for  $c > 0$ , since  $F(0)$  is negative,  $F(r)$  has to be negative definite, which requires  $\lambda$  to be negative, and (ii) for  $c < 0$ ,  $\lambda$  can be either positive or negative, making  $F(r)$  either positive definite or negative definite, respectively.

The functional form of  $F(r)$  described by the above five properties along with Eqs. (53) and (54) will yield the functional form of the metric functions  $A(r)$  and  $B(r)$ .

### B. Metric functions $A(r)$ and $B(r)$

With the boundary condition  $A(\infty) = B(\infty) = 1$ , Eqs. (53) and (54) yield

$$A(r) = \exp\left(-\int_r^\infty \frac{2c}{rF(r)} dr\right), \quad (58)$$

$$B(r) = 1 + 2(1 + c)/F(r) + c/F^2(r). \quad (59)$$

The following observations summarize the qualitative behavior of  $A(r)$  and  $B(r)$ .

- (1) From the asymptotic behavior of  $F(r) \rightarrow \lambda r$  for large  $r$ , we note that

$$A(r) = 1 - \frac{2c\lambda^{-1}}{r} + \mathcal{O}(1/r^2), \quad (60)$$

$$B(r) = 1 + \frac{2(1 + c)\lambda^{-1}}{r} + \mathcal{O}(1/r^2). \quad (61)$$

Hence, when  $|c| \gg 1$  and  $r \rightarrow \infty$ , the above solutions approximate the Schwarzschild solution with mass  $M \equiv [c\lambda^{-1}]$ . When both  $c$  and  $\lambda$  are either positive or negative, the gravity is attractive, while when one is positive and the other is negative, the gravity is repulsive.

- (2) Since  $F(r)$  is either positive definite or negative definite, both  $A(r)$  and  $B(r)$  are finite and positive for all  $r > 0$ . At  $r = 0$ , since  $F(0)$  is either  $\gamma(-\beta - 1)$  or  $\gamma(-\beta + 1)$ , Eq. (59) implies  $B(0) = 0$ .
- (3) As  $r \rightarrow 0$ , the integral in Eq. (58) diverges as  $[2c/F(0)] \ln(r)$ . When  $[c/F(0)]$  is positive, then  $A(0) = 0$ , and when  $[c/F(0)]$  is negative,  $A(0) = \infty$ . The sign of  $[c/F(0)]$  is the same as the sign of  $M = [c\lambda^{-1}]$ . For  $M > 0$ ,  $A(r)$  monotonically increases from  $A(0) = 0$  to  $A(\infty) = 1$ ; for  $M < 0$ ,  $A(r)$  monotonically decreases from  $A(0) = +\infty$  to  $A(\infty) = 1$ .
- (4) Irrespective of the sign of  $M$ ,  $B(0) = 0$  and  $B(\infty) = 1$ . However,  $B(r)$  is not necessarily monotonic. From Eq. (59), we see that  $B' = 0$  when either  $F' = 0$  or when  $F(r) = -c/(1 + c)$ . From the previous subsection,  $F' \neq 0$  for all  $r > 0$ , but  $F(r)$  could attain the value  $-c/(1 + c)$  for certain values of  $c$  and  $\lambda$ . Since  $F(r)$  is a nonvanishing monotonic function taking all values from  $F(0)$  to  $\pm\infty$ , it is straightforward to check if it would attain the value  $-c/(1 + c)$ . When  $\lambda < 0$ ,  $F(r)$  is negative definite, and  $-c/(1 + c)$  needs to be a negative number less than  $F(0) = \gamma(-\beta - 1)$ , which happens only when  $c < -1$ . When  $\lambda > 0$ ,  $F(r)$  is positive definite, and  $-c/(1 + c)$  needs to be a positive number greater than  $F(0) = \gamma(-\beta + 1)$ , which happens only when  $-1 < c < 0$ .

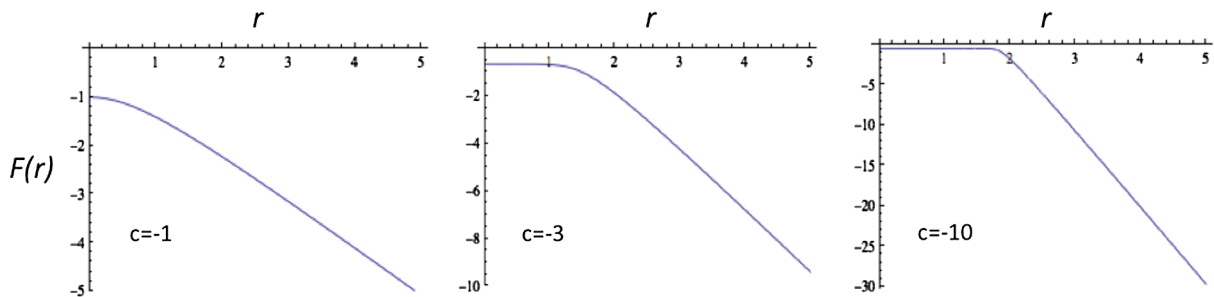


FIG. 4 (color online). Numerical solution to  $F(r)$  with  $\lambda = -1$ .

The qualitative behavior of the functions  $A(r)$  and  $B(r)$  for the various allowed ranges of  $c$  and  $\lambda$  are shown in Fig. 5. When  $M = [c\lambda^{-1}]$  is positive,  $A(r)$  is a monotonically increasing function leading to attractive gravity, and it corresponds to a Schwarzschild solution at large  $r$  when  $|c| \gg 1$ .

### C. Naked singularity at $r = 0$

An important point to note from Fig. 5 is that these solutions do not have an event horizon because both  $A(r)$  and  $B(r)$  are finite and positive for all  $r > 0$ .

Clearly, these solutions are smooth for all  $r > 0$ . However, the point  $r = 0$  is a physical singularity. Explicit calculations show that the Ricci scalar  $R^\mu_\mu$  vanishes everywhere, but the quantity  $R_{\mu\nu}R^{\mu\nu}$  is nonvanishing. It turns out that

$$R_{\mu\nu}R^{\mu\nu} = \frac{2(3F^2(r) + 2cF(r) + c^2)}{F^4(r)B^2(r)r^4}. \quad (62)$$

At  $r = 0$ , the numerator does not generally vanish, but the denominator vanishes, making  $R_{\mu\nu}R^{\mu\nu}$  diverge. Hence these solutions correspond to a naked singularity at  $r = 0$  with no event horizon to censor it.

## VII. SUMMARY AND DISCUSSION

The metric and torsion are two independent constituents of metric compatible Riemannian geometry. Because of the immense successes of torsion-free GR, torsion has not played a significant role in theories of gravity. However, when gravity is to be included with other interactions of elementary particles with intrinsic spin, a more general theory including torsion becomes imperative [3].

In the present work, torsion is incorporated in a novel way in higher dimensional Kaluza-Klein-type theories. Here torsion is not an independent degree of freedom coupled to spin; rather, it is determined in terms of the metric through a set of physically motivated constraints, which serve (i) to confine torsion to the extra dimension, leaving the 4D space-time torsion-free, and (ii) to ensure that geodesic motions in 4D remain totally unaffected by

the presence of the extra dimension. These constraints have previously been imposed in terms of veilbeins [17,19], but here it is realized that they essentially impose the requirement that the fifth dimension is hidden at the level of geodesic motion. It turns out that the nonvanishing torsion components are functions of the 5D metric components with the 4D metric  $g_{\mu\nu}$  obeying the so-called cylindrical condition; namely, it is independent of  $x^5$ . In the resulting geometry, all the 4D hypersurfaces are equivalent, and the 4D components of the connection and the Ricci tensor exactly match those of the standard 4D GR. Hence, at the level of geodesics, this geometry is virtually indistinguishable from that of the standard GR.

To proceed further, we derive modified Einstein equations from the action principle with the Ricci scalar as the Lagrangian density. In this respect, an alternate approach presents itself. In the Palatini formulation of GR, the action is varied independently with respect to the connection, and in the absence of torsion, the metricity follows from the equations of motion. Recently [20,21], it has been shown that even without assuming the absence of torsion, variations of the action independently with respect to the metric and the connection lead to GR equations along with metricity, provided the matter Lagrangian is not coupled to the connection. In our case, with the constraints imposed on the connection, it is more convenient and natural to impose metricity prior to action variation. Since the entire connection is determined to be a function of the metric, we only need to vary the action with respect to the metric to obtain the modified equations, making the theory a purely metric theory of gravity. However, adopting a Palatini-style approach, one could relax the assumption of metricity and vary the action independently with respect to the metric and the connection along with the imposed constraints, which might lead to a different set of modified Einstein equations.

We apply the ensuing modified Einstein equations to study the cosmology of a homogeneous-isotropic universe. In the matter dominated phase of the universe (zero pressure), we obtain a second order differential equation for the scale factor  $a(t)$ , in contrast to the first order differential equation in the usual FRW cosmology. In FRW cosmology,

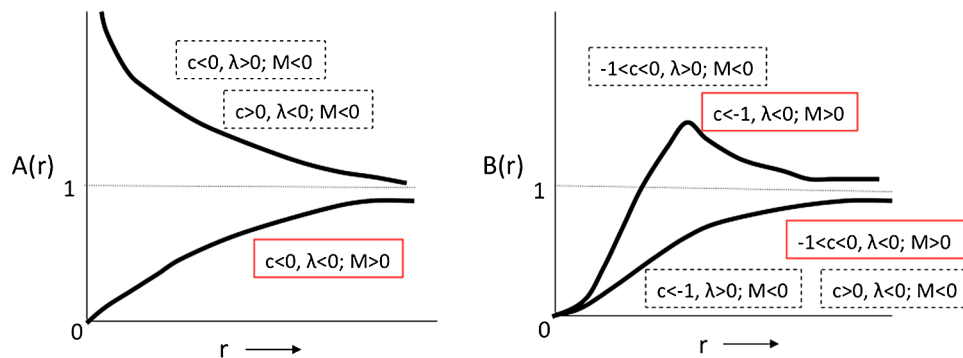


FIG. 5 (color online). Schematic behavior of  $A(r)$  and  $B(r)$ .

the second derivative of  $a(t)$  cannot be independently prescribed as an initial condition, and decelerating expansion is a necessary outcome in the absence of a cosmological constant. However, in our case, we have the choice of an initial condition for the second derivative of  $a(t)$ , which can be tuned to fit the observed acceleration of the universe. Figure 2 presents the behavior of  $a(t)$  for various choices of the current acceleration.

In an earlier version of the present work [22], the field equations were derived differently: by varying the action with respect to the metric prior to expressing the Ricci tensor in terms of the metric. In the cosmology equations generated from those field equations, acceleration was not an independently prescribable initial condition. Chen and Jing [23] showed that those equations yield accelerating universe solutions without resorting to dark energy, and that the model not only fits the supernovae data, but also solves the cosmic age problem of old high redshift objects [24]. Whether the cosmological solutions described in the current work would fit empirical results just as well needs to be investigated.

In the case of spherically symmetric vacuum solutions to the modified Einstein equations, we find some remarkably interesting results. As is well known, in the unique Schwarzschild solution of GR, when the mass is positive, an event horizon censors the central singularity. In contrast, we find positive mass naked singularity solutions that lack an event horizon. Recently, similar positive mass solutions without a horizon have been found [25] in a simpler setting of torsion-free GR with multiple extra dimensions. It would be interesting to see how the particular vacuum solutions in the torsion induced geometry in the present case match with those solutions in the torsion-free geometry.

The existence of positive mass solutions with a naked singularity have immediate consequences on gravitational collapse, opening up the possibility of an arbitrarily large star collapsing to an arbitrarily small nonsingular state. Since trapped surfaces would not necessarily form in such collapses, finite matter pressure could be sufficient to withstand a total collapse to a singularity. This suggests a detailed analysis of such solutions by treating  $-H_{\mu}{}^{\nu}$  in Eq. (27) as extra-dimensionally induced matter in standard GR. It then raises the possibility of a sufficiently strong gravitational collapse that stops short of collapsing to a singularity with a finite induced stress-energy tensor that potentially violates the weak energy condition in the region near the center. It remains to be seen if an arbitrarily small static model star with a finite stress-energy tensor can be constructed with the external geometry matching the type of solutions discussed in this paper.

In conclusion, inclusion of the torsion in the context of extra dimensions presents a novel way of obtaining modified Einstein equations that have significant physical consequences. For clarity and simplicity, we have confined the treatment to five dimensions. However, the framework can

be generalized to arbitrary dimensions  $D$ , producing a torsion-free  $D - 1$ -dimensional metric theory. Further, we could consider multiple extra dimensions and generalize the constraints so as to hide all the extra dimensions and confine the torsion in them, which would lead to a more general theory.

## ACKNOWLEDGMENTS

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## APPENDIX A: COMPUTING THE GEOMETRIC QUANTITIES IN THE 5D MANIFOLD

In this appendix, expressions for the torsion, connection coefficients, and the Ricci tensor of the 5D geometry are computed in terms of the metric. It will be shown that the physical constraints imposed on the connection will be sufficient to determine uniquely all the components of torsion and hence the other geometric quantities in terms of the metric.

### 1. 5D Levi-Civita connection

The 5D Levi-Civita connection is given by

$$\hat{\Gamma}^j{}_{ik} = \frac{1}{2}\mathbf{g}^{jm}(\partial_k\mathbf{g}_{im} + \partial_i\mathbf{g}_{km} - \partial_m\mathbf{g}_{ik}). \quad (\text{A1})$$

Expressing the 5D metric in terms of the 4D metric and the extra-dimensional metric fields given by Eq. (3), we find the 5D Levi-Civita connection to be

$$\begin{aligned} \hat{\Gamma}^5{}_{\mu\nu} &= \frac{1}{2}(\nabla_{\mu}A_{\nu} + \nabla_{\nu}A_{\mu}) + (A_{\mu}J_{\nu} + A_{\nu}J_{\mu}) \\ &+ \frac{1}{2}\epsilon\Phi^2A^{\lambda}(A_{\mu}F_{\lambda\nu} + A_{\nu}F_{\lambda\mu}) + \epsilon\Phi^2A_{\mu}A_{\nu}A^{\lambda}J_{\lambda} \\ &- \frac{1}{2}(A^{\lambda}A_{\lambda} + \epsilon\Phi^{-2})\partial_5(\epsilon A_{\mu}A_{\nu}\Phi^2) \\ &- \frac{1}{2}(A^{\lambda}A_{\lambda} + \epsilon\Phi^{-2})\partial_5g_{\mu\nu}, \end{aligned} \quad (\text{A2})$$

$$\hat{\Gamma}^{\mu}{}_{.55} = -\epsilon\Phi^2J^{\mu} + \epsilon\Phi^2\partial_5A^{\mu} + \epsilon\Phi^2A^{\mu}J_5 - \epsilon\Phi^2A_{\sigma}\partial_5g^{\mu\sigma}, \quad (\text{A3})$$

$$\hat{\Gamma}^5{}_{.55} = \epsilon\Phi^2A^{\lambda}(J_{\lambda} - \partial_5A_{\lambda}) - \epsilon\Phi^2A^{\lambda}A_{\lambda}J_5 + J_5, \quad (\text{A4})$$

$$\begin{aligned} \hat{\Gamma}^{\mu}{}_{.5\nu} &= \frac{1}{2}\epsilon\Phi^2F_{\nu}{}^{\mu} - \epsilon\Phi^2A_{\nu}J^{\mu} + \frac{1}{2}g^{\mu\sigma}\partial_5(\epsilon\Phi^2A_{\sigma}A_{\nu}) \\ &+ \frac{1}{2}g^{\mu\sigma}\partial_5g_{\nu\sigma}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \hat{\Gamma}^5{}_{.5\nu} &= \frac{1}{2}\epsilon\Phi^2A^{\lambda}F_{\lambda\nu} + \epsilon\Phi^2A_{\nu}A^{\lambda}J_{\lambda} + J_{\nu} \\ &- \frac{1}{2}A^{\lambda}\partial_5(\epsilon A_{\lambda}A_{\nu}\Phi^2) - \frac{1}{2}A^{\lambda}\partial_5g_{\nu\lambda}, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned}\hat{\Gamma}^{\lambda}_{\cdot\mu\nu} &= \Gamma^{\lambda}_{\cdot\mu\nu} + \tilde{\Gamma}^{\lambda}_{\cdot\mu\nu}, \\ \tilde{\Gamma}^{\lambda}_{\cdot\mu\nu} &= \frac{1}{2}\epsilon\Phi^2(A_{\nu}F_{\mu}^{\lambda} + A_{\mu}F_{\nu}^{\lambda}) - \epsilon\Phi^2A_{\mu}A_{\nu}J^{\lambda} \\ &\quad + \frac{1}{2}A^{\lambda}\partial_5(\epsilon A_{\mu}A_{\nu}\Phi^2) + \frac{1}{2}A^{\lambda}\partial_5g_{\mu\nu},\end{aligned}\quad (\text{A7})$$

where  $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ ,  $J_i \equiv \epsilon\Phi^{-1}\partial_i\epsilon\Phi$ ,  $\Gamma^{\lambda}_{\cdot\mu\nu}$  is the 4D Levi-Civita connection obtained from the metric  $g_{\mu\nu}$ , and  $\nabla_{\mu}$  is the derivative operator with the 4D Levi-Civita connection. The raising and lowering of indices on  $F_{\mu\nu}$ ,  $A_{\mu}$ , and  $J_{\mu}$  is performed with respect to the 4D metric.

## 2. Torsion components

We start with the conditions that  $T^{\lambda}_{\cdot\mu\nu} = 0$  and  $\tilde{\Gamma}^{\lambda}_{\cdot i5} = \tilde{\Gamma}^{\lambda}_{\cdot 5i} = 0$ . Together these conditions imply  $T^{\lambda}_{ij} = 0$ . The remaining nonvanishing components of torsion that need to be determined are  $T^5_{ij}$ , a total of ten independent components. The conditions  $\tilde{\Gamma}^{\lambda}_{\cdot 5i} = 0$ , consisting of 20 equations, are sufficient to determine uniquely all the nonvanishing torsion components.

$$\tilde{\Gamma}^{\mu}_{\cdot i5} = \hat{\Gamma}^{\mu}_{\cdot i5} + K^{\mu}_{i5} = 0. \quad (\text{A8})$$

From the above equation, the nonvanishing components of contorsion and torsion can be determined in terms of components of the 5D Levi-Civita connection  $\hat{\Gamma}_{\cdot}$ . First taking  $i = 5$ ,

$$K^{\mu}_{55} = \mathbf{g}^{\mu j}\mathbf{g}_{55}T^5_{j5} = \mathbf{g}^{\mu\nu}\mathbf{g}_{55}T^5_{\nu 5} = -\hat{\Gamma}^{\mu}_{\cdot 55}. \quad (\text{A9})$$

From the 5D metric [Eq. (2)], we have  $\mathbf{g}^{\mu\nu} = g^{\mu\nu}$ . Multiplying both sides by  $g_{\mu\sigma}$  and using the orthogonality relations of the metric, we obtain

$$\begin{aligned}g_{\mu\sigma}g^{\mu\nu}\mathbf{g}_{55}T^5_{\nu 5} &= -g_{\mu\sigma}\hat{\Gamma}^{\mu}_{\cdot 55}, \\ \mathbf{g}_{55}T^5_{\sigma 5} &= -g_{\mu\sigma}\hat{\Gamma}^{\mu}_{\cdot 55}, \\ \Rightarrow T^5_{\sigma 5} &= -g_{\mu\sigma}\hat{\Gamma}^{\mu}_{\cdot 55}\epsilon\Phi^{-2}.\end{aligned}\quad (\text{A10})$$

Substituting for  $\hat{\Gamma}^{\mu}_{\cdot 55}$  from Eq. (A3), we obtain

$$T^5_{\sigma 5} = g_{\mu\sigma}[J^{\mu} - \partial_5A^{\mu} - A^{\mu}J_5 + A_{\sigma}\partial_5g^{\mu\sigma}]. \quad (\text{A11})$$

We have thus far used four equations and solved for four of the ten independent torsion components. Next, take  $i = \nu$  in Eq. (A8). The contorsion components  $K^{\mu}_{\nu 5}$  can be expressed in terms of the torsion components as follows:

$$\begin{aligned}K^{\mu}_{\nu 5} &= \frac{1}{2}\mathbf{g}^{\mu j}(\mathbf{g}_{55}T^5_{j\nu} + \mathbf{g}_{\nu 5}T^5_{j5}) \\ &= \frac{1}{2}\mathbf{g}^{\mu 5}(\mathbf{g}_{55}T^5_{5\nu}) + \frac{1}{2}\mathbf{g}^{\mu\sigma}(\mathbf{g}_{55}T^5_{\sigma\nu} + \mathbf{g}_{\nu 5}T^5_{\sigma 5}) \\ &= \frac{1}{2}\epsilon A^{\mu}\Phi^2T^5_{\nu 5} + \frac{1}{2}\epsilon A_{\nu}\Phi^2g^{\mu\sigma}T^5_{\sigma 5} + \frac{1}{2}\epsilon\Phi^2g^{\mu\sigma}T^5_{\sigma\nu}.\end{aligned}\quad (\text{A12})$$

From  $K^{\mu}_{\nu 5} = -\hat{\Gamma}^{\mu}_{\cdot\nu 5}$ , and using Eq. (A10), we have

$$-2\hat{\Gamma}^{\mu}_{\cdot\nu 5} + A^{\mu}g_{\alpha\nu}\hat{\Gamma}^{\alpha}_{\cdot 55} + A_{\nu}\hat{\Gamma}^{\mu}_{\cdot 55} = \epsilon\Phi^2g^{\mu\sigma}T^5_{\sigma\nu} \quad (\text{A13})$$

and hence,

$$T^5_{\sigma\nu} = \epsilon\Phi^{-2}g_{\mu\sigma}[-2\hat{\Gamma}^{\mu}_{\cdot\nu 5} + A^{\mu}g_{\lambda\nu}\hat{\Gamma}^{\lambda}_{\cdot 55} + A_{\nu}\hat{\Gamma}^{\mu}_{\cdot 55}]. \quad (\text{A14})$$

There are 16 equations. Though the torsion in the left-hand side (l.h.s.) is antisymmetric, the r.h.s. is a combination of symmetric and antisymmetric terms. Substituting for the 5D Levi-Civita connection from Sec. A 1, the above equation has the form

$$\begin{aligned}T^5_{\sigma\nu} &= [\partial_{\sigma}A_{\nu} - \partial_{\nu}A_{\sigma}] + [J_{\sigma}A_{\nu} - J_{\nu}A_{\sigma}] \\ &\quad + [\epsilon\Phi^{-2}\partial_5g_{\sigma\nu} + A_{\sigma}A^{\lambda}\partial_5g_{\lambda\nu} + A_{\nu}A^{\lambda}\partial_5g_{\lambda\sigma}].\end{aligned}\quad (\text{A15})$$

The first two terms on the r.h.s. above are antisymmetric in  $\sigma$  and  $\nu$ , while the third term is symmetric. The antisymmetry of torsion implies that the symmetric terms on the r.h.s. must be zero, and hence

$$\partial_5g_{\sigma\nu} = 0. \quad (\text{A16})$$

Consequently,

$$T^5_{\sigma\nu} = [\partial_{\sigma}A_{\nu} - \partial_{\nu}A_{\sigma}] + [J_{\sigma}A_{\nu} - J_{\nu}A_{\sigma}]. \quad (\text{A17})$$

Thus the 20 equations of the imposed condition [Eq. (A8)] have determined all ten independent nonvanishing components of the torsion [Eqs. (A11) and (A17)] and, in addition, have imposed a constraint on the ten independent components of the 4D metric  $g_{\sigma\nu}$ , making them independent of  $x^5$ .

## 3. The contorsion, connection, and Ricci tensor

Some components of contorsion are directly prescribed by the imposed condition, namely,  $K^{\mu}_{i5} = K^{\mu}_{5i} = -\hat{\Gamma}^{\mu}_{\cdot i5}$ . The remaining components of the contorsion can be calculated from the torsion components by using Eq. (7). Since  $T^5_{ij}$  are the only nonvanishing components of torsion, it follows that

$$\begin{aligned}K^{\mu}_{\cdot i5} &= K^{\mu}_{5i} = -\hat{\Gamma}^{\mu}_{\cdot i5}, \\ K^5_{i5} &= \frac{1}{2}T^5_{i5} + \frac{1}{2}\mathbf{g}^{5j}(\mathbf{g}_{i5}T^5_{j5} + \mathbf{g}_{55}T^5_{ji}), \\ K^5_{5i} &= K^5_{i5} + T^5_{5i}, \\ K^{\mu}_{\nu\lambda} &= \frac{1}{2}\mathbf{g}^{\mu j}(\mathbf{g}_{\nu 5}T^5_{j\lambda} + \mathbf{g}_{\lambda 5}T^5_{j\nu}), \\ K^5_{\nu\lambda} &= \frac{1}{2}T^5_{\nu\lambda} + \frac{1}{2}\mathbf{g}^{5j}(\mathbf{g}_{\nu 5}T^5_{j\lambda} + \mathbf{g}_{\lambda 5}T^5_{j\nu}).\end{aligned}\quad (\text{A18})$$

Substituting for the nonvanishing torsion components given by Eqs. (A11) and (A17), along with the requirement that the 4D metric is independent of the fifth dimension [Eq. (A16)], the components of contorsion are found to be

$$\begin{aligned}
K_{\cdot i5}^\mu &= K_{\cdot 5i}^\mu = -\tilde{\Gamma}_{\cdot i5}^\mu, \\
K_{\cdot i5}^5 &= -\tilde{\Gamma}_{\cdot i5}^5 + J_i, \quad K_{\cdot 5i}^5 = K_{\cdot i5}^5 + T_{\cdot 5i}^5, \\
K_{\cdot \nu\lambda}^\mu &= \epsilon \Phi^2 F_{(\lambda}^\mu A_{\nu)} + A_\lambda A_\nu J^\mu \epsilon \Phi^2 - \frac{1}{2} A^\mu \partial_5 (\epsilon A_\nu A_\lambda \Phi^2), \\
K_{\cdot \nu\lambda}^5 &= -\epsilon \Phi^2 A^\sigma F_{\sigma(\lambda} A_{\nu)} - (A_\sigma J^\sigma) \epsilon A_\nu A_\lambda \Phi^2 \\
&\quad + \frac{1}{2} (A_\sigma A^\sigma + \epsilon \Phi^{-2}) \partial_5 (\epsilon A_\nu A_\lambda \Phi^2) - A_\nu J_\lambda + \frac{1}{2} F_{\nu\lambda}.
\end{aligned} \tag{A19}$$

Now from Eq. (5), we obtain all the connection coefficients,

$$\begin{aligned}
\tilde{\Gamma}_{\cdot \mu\nu}^5 &= \nabla_\mu A_\nu + J_\mu A_\nu, \\
\tilde{\Gamma}_{\cdot 5\mu}^5 &= \partial_5 A_\mu + J_5 A_\mu, \\
\tilde{\Gamma}_{\cdot \mu 5}^5 &= J_\mu, \quad \tilde{\Gamma}_{\cdot 55}^5 = J_5, \\
\tilde{\Gamma}_{\cdot \mu\nu}^\lambda &= \Gamma_{\cdot \mu\nu}^\lambda, \\
\tilde{\Gamma}_{\cdot 55}^\mu &= \tilde{\Gamma}_{\cdot \nu 5}^\mu = \tilde{\Gamma}_{\cdot 5\nu}^\mu = 0.
\end{aligned} \tag{A20}$$

Taking  $i = \mu$  and  $k = \nu$  in the Ricci tensor defined by Eq. (15), we have

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= +\partial_\nu \tilde{\Gamma}_{\cdot \sigma\mu}^\sigma - \partial_\sigma \tilde{\Gamma}_{\cdot \nu\mu}^\sigma + \tilde{\Gamma}_{\cdot \nu\lambda}^\sigma \tilde{\Gamma}_{\cdot \sigma\mu}^\lambda - \tilde{\Gamma}_{\cdot \sigma\lambda}^\sigma \tilde{\Gamma}_{\cdot \nu\mu}^\lambda \\
&\quad + \partial_\nu \tilde{\Gamma}_{\cdot 5\mu}^5 - \partial_5 \tilde{\Gamma}_{\cdot \nu\mu}^5 + \tilde{\Gamma}_{\cdot \nu\lambda}^5 \tilde{\Gamma}_{\cdot 5\mu}^\lambda - \tilde{\Gamma}_{\cdot 5\lambda}^5 \tilde{\Gamma}_{\cdot \nu\mu}^\lambda \\
&\quad + \tilde{\Gamma}_{\cdot \nu 5}^\sigma \tilde{\Gamma}_{\cdot \sigma\mu}^5 - \tilde{\Gamma}_{\cdot \sigma 5}^\sigma \tilde{\Gamma}_{\cdot \nu\mu}^5 + \tilde{\Gamma}_{\cdot \nu 5}^5 \tilde{\Gamma}_{\cdot 5\mu}^5 - \tilde{\Gamma}_{\cdot 55}^5 \tilde{\Gamma}_{\cdot \nu\mu}^5.
\end{aligned} \tag{A21}$$

Since  $\tilde{\Gamma}_{\cdot}$  is the same as  $\Gamma_{\cdot}$  when all the indices are four dimensional, the first line is clearly the 4D Ricci tensor. The terms in the subsequent lines can be reexpressed in terms of the 4D covariant derivative operator as follows:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\nu \tilde{\Gamma}_{\cdot 5\mu}^5 - \partial_5 \tilde{\Gamma}_{\cdot \nu\mu}^5 - \tilde{\Gamma}_{\cdot 55}^5 \tilde{\Gamma}_{\cdot \nu\mu}^5 + \tilde{\Gamma}_{\cdot \nu 5}^5 \tilde{\Gamma}_{\cdot 5\mu}^5. \tag{A22}$$

Substituting for the connection  $\tilde{\Gamma}_{\cdot}$  from Eq. (A21), we find after some algebra this can be simplified as

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= R_{\mu\nu} + \nabla_\nu (\partial_5 A_\mu + J_5 A_\mu) - \partial_5 (\nabla_\nu A_\mu + J_\nu A_\mu) \\
&\quad - J_5 (\nabla_\nu A_\mu + J_\nu A_\mu) + J_\nu (\partial_5 A_\mu + J_5 A_\mu) \\
&= R_{\mu\nu} + A_\lambda \partial_5 \Gamma_{\cdot \mu\nu}^\lambda \\
&= R_{\mu\nu}.
\end{aligned} \tag{A23}$$

Similarly, the other components of the Ricci tensor are found to be

$$\tilde{R}_{\mu 5} = \partial_5 \tilde{\Gamma}_{\cdot \sigma\mu}^\sigma = 0, \tag{A24}$$

$$\tilde{R}_{5\nu} = \partial_\nu \tilde{\Gamma}_{\cdot 55}^5 - \partial_5 \tilde{\Gamma}_{\cdot \nu 5}^5 = 0, \tag{A25}$$

$$\tilde{R}_{55} = 0. \tag{A26}$$

Note that neither the connection nor the Ricci tensor depends on the signature ( $\epsilon$ ) of the fifth dimension.

## APPENDIX B: COMPUTING THE MODIFIED EINSTEIN TENSOR

Here we provide some intermediate steps to go from Eqs. (24) and (25) to Eq. (26) and obtain a simplified expression for  $H_{\mu\nu}$ . With  $B_\nu \equiv T_{\cdot \nu 5}^5$  and  $H_{\mu\nu}$  defined by

$$\begin{aligned}
&\int B_\nu [g^{\mu\alpha} \delta \Gamma_{\cdot \mu\alpha}^\nu - g^{\mu\nu} \delta \Gamma_{\cdot \alpha\mu}^\alpha] \sqrt{-\mathbf{g}} d^5 x \\
&= \int H_{\mu\nu} \delta g^{\mu\nu} \sqrt{-\mathbf{g}} d^5 x,
\end{aligned} \tag{B1}$$

we show that

$$H_{\mu\nu} = \nabla_{(\mu} B_{\nu)} - (\nabla \cdot B) g_{\mu\nu} + J_{(\mu} B_{\nu)} - (J \cdot B) g_{\mu\nu}. \tag{B2}$$

*Proof.*—Consider the first term in the integrand of the l.h.s. of Eq. (B1),

$$\begin{aligned}
2g^{\mu\alpha} \delta \Gamma_{\cdot \mu\alpha}^\nu &= g^{\mu\alpha} g^{\nu\lambda} (\partial_\mu \delta g_{\lambda\alpha} + \partial_\alpha \delta g_{\lambda\mu} - \partial_\lambda \delta g_{\mu\alpha}) \\
&\quad + g^{\mu\alpha} (\partial_\mu g_{\lambda\alpha} + \partial_\alpha g_{\lambda\mu} - \partial_\lambda g_{\mu\alpha}) \delta g^{\nu\lambda}.
\end{aligned} \tag{B3}$$

The variations of the covariant metric in the above equation can be reexpressed in terms of the variations of the contravariant metric using the identity  $\delta g_{\alpha\lambda} = -g_{\alpha\mu} g_{\lambda\nu} \delta g^{\mu\nu}$ . We note that

$$\begin{aligned}
g^{\mu\alpha} g^{\nu\lambda} \partial_\mu \delta g_{\lambda\alpha} &= -g^{\nu\lambda} \partial_\mu g_{\lambda\sigma} (\delta g^{\sigma\mu}) \\
&\quad - g^{\mu\alpha} \partial_\mu g_{\beta\alpha} (\delta g^{\nu\beta}) - \partial_\mu (\delta g^{\nu\mu}), \\
g^{\mu\alpha} g^{\nu\lambda} \partial_\alpha \delta g_{\lambda\mu} &= g^{\mu\alpha} g^{\nu\lambda} \partial_\mu \delta g_{\lambda\alpha}, \\
-g^{\mu\alpha} g^{\nu\lambda} \partial_\lambda \delta g_{\mu\alpha} &= g^{\nu\lambda} \partial_\lambda g_{\mu\sigma} (\delta g^{\sigma\mu}) + g^{\nu\lambda} \partial_\lambda g_{\beta\sigma} (\delta g^{\sigma\beta}) \\
&\quad + g^{\nu\lambda} g_{\beta\sigma} \partial_\lambda (\delta g^{\sigma\beta}).
\end{aligned}$$

Taken together, we obtain

$$\begin{aligned}
2g^{\mu\alpha} \delta \Gamma_{\cdot \mu\alpha}^\nu &= -2g^{\nu\lambda} \partial_\mu g_{\lambda\sigma} (\delta g^{\sigma\mu}) - 2\partial_\mu (\delta g^{\nu\mu}) \\
&\quad + 2g^{\nu\lambda} \partial_\lambda g_{\mu\sigma} (\delta g^{\sigma\mu}) + g^{\nu\lambda} g_{\beta\sigma} \partial_\lambda (\delta g^{\sigma\beta}) \\
&\quad - g^{\mu\alpha} \partial_\lambda g_{\mu\alpha} (\delta g^{\nu\lambda}).
\end{aligned} \tag{B4}$$

Next consider the second term in the integrand of the l.h.s. of Eq. (B1). Since  $2\Gamma_{\cdot \alpha\mu}^\alpha = g^{\alpha\lambda} \partial_\mu g_{\alpha\lambda}$ ,

$$\begin{aligned}
2\delta \Gamma_{\cdot \alpha\mu}^\alpha &= (\delta g^{\alpha\lambda}) \partial_\mu g_{\alpha\lambda} + g^{\alpha\lambda} \partial_\mu (\delta g_{\alpha\lambda}) \\
&= -(\delta g^{\alpha\lambda}) \partial_\mu g_{\alpha\lambda} - g_{\alpha\lambda} \partial_\mu (\delta g^{\alpha\lambda}).
\end{aligned} \tag{B5}$$

Using Eqs. (B4) and (B5), we find

$$\begin{aligned}
&B_\nu [g^{\mu\alpha} \delta \Gamma_{\cdot \mu\alpha}^\nu - g^{\mu\nu} \delta \Gamma_{\cdot \alpha\mu}^\alpha] \sqrt{-\mathbf{g}} \\
&= B_\nu [-\partial_\mu (\delta g^{\mu\nu}) + g^{\mu\nu} g_{\alpha\beta} \partial_\mu (\delta g^{\alpha\beta})] \sqrt{-\mathbf{g}} \\
&\quad + B_\nu [-g^{\nu\lambda} \partial_\mu g_{\lambda\sigma} (\delta g^{\sigma\mu}) + \frac{3}{2} g^{\nu\lambda} \partial_\lambda g_{\mu\sigma} (\delta g^{\sigma\mu}) \\
&\quad - \frac{1}{2} g^{\mu\sigma} \partial_\lambda g_{\mu\sigma} (\delta g^{\nu\lambda})] \sqrt{-\mathbf{g}}.
\end{aligned} \tag{B6}$$

The first term in the r.h.s. of the above equation contains terms with the derivatives of the variation. We note that these terms are eventually going to be integrated. By integrating them by parts and ignoring the boundary terms, the above equation takes the form

$$\begin{aligned}
& B_\nu [g^{\mu\alpha} \delta \Gamma_{\mu\alpha}^\nu - g^{\mu\nu} \delta \Gamma_{\alpha\mu}^\alpha] \sqrt{-\mathbf{g}} \\
&= [\partial_\mu (B_\nu \sqrt{-\mathbf{g}}) \delta g^{\mu\nu} - \partial_\mu (g^{\mu\nu} g_{\alpha\beta} B_\nu \sqrt{-\mathbf{g}}) \delta g^{\alpha\beta}] \\
&\quad + B_\nu [-g^{\nu\lambda} \partial_\mu g_{\lambda\sigma} (\delta g^{\sigma\mu}) + \frac{3}{2} g^{\nu\lambda} \partial_\lambda g_{\mu\sigma} (\delta g^{\sigma\mu}) \\
&\quad - \frac{1}{2} g^{\mu\sigma} \partial_\lambda g_{\mu\sigma} (\delta g^{\nu\lambda})] \sqrt{-\mathbf{g}}. \tag{B7}
\end{aligned}$$

To simplify the r.h.s. of Eq. (B7) it is useful to note the following identities:

$$\partial_\mu (g^{\mu\nu}) = g^{\sigma\nu} g_{\sigma\lambda} \partial_\mu (g^{\mu\lambda}) = -g^{\sigma\nu} g^{\mu\lambda} \partial_\mu (g_{\sigma\lambda}), \tag{B8}$$

$$\partial_\mu (\sqrt{-\mathbf{g}}) = \frac{\partial_\mu \mathbf{g}}{2\mathbf{g}} \sqrt{-\mathbf{g}} = \frac{\sqrt{-\mathbf{g}}}{2} \mathbf{g}^{ij} \partial_\mu (\mathbf{g}_{ij}), \tag{B9}$$

$$\begin{aligned}
\mathbf{g}^{ij} \partial_\mu (\mathbf{g}_{ij}) &= g^{\sigma\lambda} \partial_\mu (g_{\sigma\lambda}) + g^{\sigma\lambda} \partial_\mu (\epsilon A_\sigma A_\lambda \Phi^2) \\
&\quad + 2\mathbf{g}^{\sigma 5} \partial_\mu (\epsilon A_\sigma \Phi^2) + \mathbf{g}^{55} \partial_\mu (\epsilon \Phi^2) \\
&= g^{\sigma\lambda} \partial_\mu (g_{\sigma\lambda}) + 2J_\mu. \tag{B10}
\end{aligned}$$

Using the above identities, the r.h.s of Eq. (B7) becomes

$$\begin{aligned}
& B_\nu [g^{\mu\alpha} \delta \Gamma_{\mu\alpha}^\nu - g^{\mu\nu} \delta \Gamma_{\alpha\mu}^\alpha] \sqrt{-\mathbf{g}} \\
&= [\partial_\mu (B_\nu) \delta g^{\mu\nu} - \partial_\mu (B_\nu) g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} \\
&\quad + B_\nu g_{\alpha\beta} g^{\sigma\nu} g^{\mu\lambda} \partial_\mu (g_{\sigma\lambda}) \delta g^{\alpha\beta} \\
&\quad - \frac{1}{2} B_\nu g^{\mu\nu} g_{\alpha\beta} g^{\sigma\lambda} \partial_\mu (g_{\sigma\lambda}) \delta g^{\alpha\beta} \\
&\quad - B_\nu g^{\nu\sigma} \partial_\mu (g_{\sigma\alpha}) \delta g^{\alpha\mu} + \frac{1}{2} B_\nu g^{\nu\sigma} \partial_\sigma (g_{\alpha\mu}) \delta g^{\alpha\mu} \\
&\quad + B_\nu J_\mu \delta g^{\mu\nu} - B_\nu J_\mu g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta}] \sqrt{-\mathbf{g}}. \tag{B11}
\end{aligned}$$

Rewriting the derivatives of the metric in terms of the Levi-Civita connection, we find

$$\begin{aligned}
& B_\nu [g^{\mu\alpha} \delta \Gamma_{\mu\alpha}^\nu - g^{\mu\nu} \delta \Gamma_{\alpha\mu}^\alpha] \sqrt{-\mathbf{g}} \\
&= [\partial_\mu (B_\nu) \delta g^{\mu\nu} - \partial_\mu (B_\nu) g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} \\
&\quad - B_\nu \Gamma_{\mu\sigma}^\nu \delta g^{\sigma\mu} + B_\nu \Gamma_{\mu\sigma}^\nu g^{\sigma\mu} g_{\alpha\beta} \delta g^{\alpha\beta} \\
&\quad + B_\nu J_\mu \delta g^{\mu\nu} - B_\nu J_\mu g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta}] \sqrt{-\mathbf{g}} \\
&= [\nabla_\mu B_\nu - (\nabla \cdot B) g_{\mu\nu} + J_\mu B_\nu \\
&\quad - (J \cdot B) g_{\mu\nu}] \delta g^{\mu\nu} \sqrt{-\mathbf{g}}. \tag{B12}
\end{aligned}$$

Since the variation  $\delta g^{\mu\nu}$  is symmetric in the indices  $\mu$  and  $\nu$ , only the symmetric part of the r.h.s. of the above equation will contribute to the equations of motion. Hence  $H_{\mu\nu}$  will be given by Eq. (B2).

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