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Note on Rainbow Connection in Oriented Graphs with Diameter 2

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Abstract

In this note, we provide a sharp upper bound on the rainbow connection number of tournaments of diameter 2. For a tournament T of diameter 2, we show $2 \leq \overrightarrow{rc}(T) \leq 3$. Furthermore, we provide a general upper bound on the rainbow k-connection number of tournaments as a simple example of the probabilistic method. Finally, we show that an edge-colored tournament of k^{th} diameter 2 has rainbow k-connection number at most approximately k^2.

1 Introduction

The concept of rainbow connection was first introduced by Chartrand et al. in [3]. A path in an edge-colored graph is called *rainbow* if no two edges in the path receive the same color. The *rainbow connection number* of a graph is the minimum number of colors needed to color the edges of the graph so that there is a rainbow path between every pair of vertices. This and the more general rainbow k-connection number have been heavily studied in recent years in [2, 3, 4, 5, 7, 8, 10] and many other works. In particular, see [9] for a survey of results in the area.

A tournament T is an oriented complete graph. We consider only *k-strongly connected* (or simply *k-strong*) tournaments, meaning that there are k internally disjoint directed paths from each vertex to every other vertex. A directed path between two vertices in an edge-colored tournament is called *rainbow* if no two edges have the same color within the path. If there is a directed rainbow path between every pair of vertices in a graph, then the coloring is called *rainbow connected*. The smallest number of colors needed for a tournament to be rainbow connected is called the (directed) *rainbow connection number*, denoted $\overrightarrow{rc}(T)$. The diameter d of a tournament is the largest, over all ordered pairs of vertices, number of edges in the shortest path between the two vertices.

In [6], the following theorem was proven.

Theorem 1 (Dorbec et al. [6]). *For any tournament T of diameter d,*

$$d \leq \overrightarrow{rc}(T) \leq d + 2.$$

The authors noted that $d + 2$ may not be the best upper bound.

Question 1. *For each diameter d, is $d + 1$ or $d + 2$ the sharp upper bound on $\overrightarrow{rc}(T)$ where T has diameter d.*

We believe that a $(d + 1)$-coloring is possible, at least in some cases. Indeed, we show that for tournaments of diameter 2, this improved upper bound holds.

Theorem 2. *For any tournament T of diameter 2,*

$$2 \leq \overrightarrow{rc}(T) \leq 3.$$

The proof of this result is provided in Section 2.

More generally, we initiate the study of the rainbow k-connection number of a tournament. An edge-colored tournament is called *rainbow k-connected* if, between every pair of vertices, there is a set of k internally disjoint rainbow paths. The *rainbow k-connection number* of a tournament, denoted $\overrightarrow{rc}_k(T)$, is then the minimum number of colors needed to produce a rainbow k-connected coloring of the tournament T. To state our next result, we let the *k-total-diameter*, denoted $d_k(T)$, be the maximum (over all pairs of vertices) of the smallest number of edges in a set of k internally disjoint paths between the vertices.

Theorem 3. *Given an integer $k \geq 2$ and a tournament T of order n with $d_k(T) = d$,*

$$\overrightarrow{rc}_k(T) \leq \frac{d}{1-(1-\frac{1}{n^2})^{1/d}}.$$

The proof of Theorem 3 is an easy application of the probabilistic method and is presented in Section 3.

Next we define some more notation. Say a set of k internally disjoint paths from a vertex x to a vertex y is *minimum* if the longest path in the set is as short as possible, over all such sets of paths. Let the *k^{th} diameter* denote the maximum length, over all pairs of vertices u, v, of the longest path in a minimum set of k internally disjoint $u-v$ paths. More formally, if $\ell_k(u,v)$ is the minimum length of the longest path in a set of k internally disjoint $u-v$ paths, then the k^{th} diameter of a graph G is

$$\max_{u,v\in V(G)} \ell_k(u,v).$$

Note that the 1^{st} diameter is simply the diameter of the graph. Also note that the k^{th} diameter is at least $\frac{d_k(T)}{k}$. Our final result considers tournaments with small k^{th} diameter and provides a bound on the rainbow connection number.

Theorem 4. *A strongly connected tournament T of k^{th} diameter 2 has $\overrightarrow{rc}_k(T) \leq 3+k+2\binom{k}{2}$.*

The proof of Theorem 4 is presented in Section 4. This naturally leads to the following problem.

Problem 1. *Produce sharp bounds on $\overrightarrow{rc}_k(T)$ in terms of the k^{th} diameter of T.*

2 Proof of Theorem 2

The sharpness of the upper bound is given by the following example. Let A be a directed triangle and let u and v be single vertices. Direct all edges from v to A, from A to u and from u to v. Any 2-coloring of this graph must color two edges of A with a single color. This induces a directed monochromatic P_3. Let a_1 be the initial vertex of this P_3 and let a_3 be the terminal vertex and note that the edge between a_1 and a_3 is directed from a_3 to a_1. See Figure 1.

The only possible rainbow path from a_1 to a_3 must pass through u and v, meaning that it must use 3 different colors. Thus, this graph has diameter 2 but rainbow connection number 3. Larger graphs with the same property can be built by replacing vertices with directed triangles and blowing up edges in the natural way.

We now prove Theorem 2.

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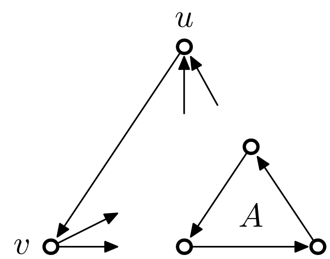

Figure 1: A tournament T with diameter 2 and rainbow connection number 3.

Proof. Let T be a tournament of diameter 2. Let $a \to b \to c$ be a shortest path from a vertex a to another vertex c. Let A_1 denote the out-neighborhood of a and let $A_2 = T \setminus (A_1 \cup \{a\})$. Note that A_i is the set of vertices at distance i from a and, in particular, $b \in A_1$ and $c \in A_2$. Color all edges from a to A_1 with color 1 (red in Figure 2). All edges from vertices in A_1 to vertices in A_2 have color 2 (blue), and all edges from vertices in A_2 to the vertex a have color 3 (green). All edges from vertices in A_2 to vertices in A_1 also have color 3. Finally, all edges within the same set, either A_1 or A_2, have color 1. This coloring is similar to the one used by Dorbec et al. in [6] to prove Theorem 1.

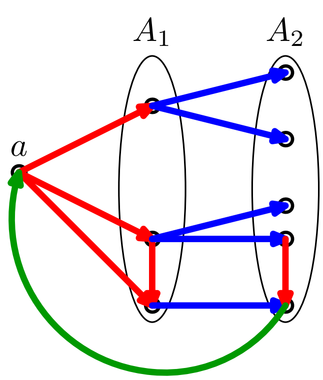

Figure 2: Coloring of the tournament.

In order to show that this coloring is rainbow connected, we consider cases based on the location of two selected vertices x and y and find rainbow paths between them.

If $x = a$, we trivially find a rainbow path to y for any choice of $y \in A_1$ since A_1 is the out-neighborhood of a. If $y \in A_2$, then by construction, there is a rainbow path containing some vertex $w \in A_1$ such that $x \to w \to y$ with colors 1 and 2 respectively.

If $y = a$ and $x \in A_2$, the result is again trivial since a is an out-neighbor of every vertex in A_2. Also, if $x \in A_1$, then again there is a rainbow path of length 2, namely $x \to w \to y$ for some $w \in A_2$ using colors 2 and 3.

If $x \in A_2$ and $y \in A_1$, then, by construction, there is a rainbow path $x \to a \to y$ using colors 3 and 1, respectively.

Finally, suppose $x \in A_1$ and $y \in A_2$. If the edge $x \to y$ is in $E(T)$, then there is a trivially rainbow path of length 1. Since the diameter is 2, there exists a vertex with $x \to w \to y$. Regardless of the location of w, this path is rainbow by construction. More specifically, if

$w \in A_1$, then the path $x \to w \to y$ uses colors 1 and 2, respectively. If $w \in A_2$, then the path $x \to w \to y$ exists uses colors 2 and 1, respectively. Hence, every tournament with diameter 2 has rainbow connection number at most 3. $\square$

3 Proof of Theorem 3

For this proof, we use the probabilistic method as described by Alon and Spencer in [1]. This bound is likely far from the best possible, particularly when n is much bigger than d, and we make little effort to optimize it.

We now prove Theorem 3.

Proof. Consider a tournament on n vertices and set $c = \frac{d}{1-(1-\frac{1}{n^2})^{1/d}}$. Label the vertices of T with $\{v_1, v_2, \ldots, v_n\}$. Randomly color the edges of T using c colors. Let $X_{i,j}$ be an indicator variable which takes the value 1 if there is no set of k internally disjoint rainbow paths from v_i to v_j. Since there is a set of such paths on at most d edges, we compute the expectation of $X_{i,j}$ to be

$$E(X_{i,j}) = 1 - \frac{c!}{(c-d+1)!c^d} < 1 - \left(1 - \frac{d}{c}\right)^d.$$

By linearity of expectation, if we set $X = \sum_{i,j} X_{i,j}$, we get

$$E(X) = \sum E(X_{i,j}) \leq 2\binom{n}{2}\left(1 - \left(1 - \frac{d}{c}\right)^d\right).$$

Since $c = \frac{d}{1-(1-\frac{1}{n^2})^{1/d}}$, we see that $E(X) < 1$ so, by the probabilistic method, there is a coloring of T with c colors that is rainbow k-connected. $\square$

4 Proof of Theorem 4

For a tournament T of k^{th} diameter 2, we use the following coloring. Select a k-subset of vertices $A := \{v_1, v_2, \ldots, v_k\}$. Now, let the set of all out-neighbors of A be called A_1 and color all edges of the form $A \to A_1$ with the color C_1 and edges of the form $A_1 \to A$ with color C_2. Let $N'_i \subseteq A_1$ be the out-neighborhood of v_i for each $1 \leq i \leq k$. Define sets

$$N_i = N'_i \setminus (\cup_{j<i} N_j)$$

for all $1 \leq i \leq k$. We use one distinct color C_{N_i} on all edges within each set N_i for all i for a total of k colors. Use at most $2\binom{k}{2}$ distinct colors to color the remaining edges of A_1 such that edges of the form $N_i \to N_j$ have a different color from those of the form $N_j \to N_i$ for all $i \neq j$. This uses a total of $2\binom{k}{2} + k$ colors to color A_1.

Let A_2 be the set of all remaining vertices so that A_2 is in the out-neighborhood of A_1 and A is in the out-neighborhood of A_2. It should be noted that $A_2 = \emptyset$ is allowed. Color all edges of the form $A_1 \to A_2$ with color C_2 and the edges from $A_2 \to A$ with color C_3. All edges from A to A_2 have color C_1 and all edges from A_2 to A_1 have color C_3. The edges within the set A are colored using $\binom{k}{2}$ of the colors previously used between sets in A_1. The edges within the set A_2, if they exist, are allowed to be any color except C_3. Hence $3 + k + 2\binom{k}{2}$ colors are used to color T.

In order to show that this coloring is rainbow connected, we consider cases based on the location of vertices x and y and find k rainbow paths from x to y.

If $x = v_i$ and $y = v_j$ for some $i \neq j$, then there are k internally disjoint paths of length at most 2 that each must be one the following: $x \to y$, $x \to N_i \to y$, or $x \to v_h \to y$ for some $h \neq i, j$. All paths of these forms are rainbow by construction.

If $x, y \in A_2$ then there are k internally disjoint paths of length 3 that each must be one of the following: $x \to v_i \to N_i \to y$, or $x \to v_i \to N_j \to y$ for all i with $1 \leq i \leq k$ and for some $j \neq i$. All such paths are rainbow by construction.

If $x \in A_1$ and $y = v_i$ then there are k internally disjoint rainbow paths of length at most 2, each having one the following forms: $x \to v_k \to y$, or $x \to w \to y$, where $w \in A_1$ and $w = y$ is allowed. If $x = v_i$ and $y \in A_1$ then there exist k internally disjoint rainbow paths of the form $x \to w \to y$ where each $w \in A_1$ and $w = y$ is allowed.

If $x = v_i$ and $y \in A_2$ then there are k internally disjoint rainbow paths of the form $x \to A_1 \to y$. If $x \in A_2$ and $y = v_i$ then there exist k internally disjoint rainbow paths $x \to w \to y$ where each w is anywhere and $w = y$ is allowed.

If $x \in A_1$, say $x \in N_i$, and $y \in A_2$, then there are easily k internally disjoint rainbow paths from x to y as in previous cases. If $x \in A_2$ and $y \in A_1$ then there are k internally disjoint rainbow paths such that $x \to v_i \to N_i \to y$ for all i.

Finally, suppose $x, y \in A_1$. If $x \in N_i$ and $y \notin N_i$, then there are k internally disjoint rainbow paths of the form $x \to w \to y$, where w is anywhere. Now let $x, y \in N_i$. Since there are k internally disjoint paths of length at most 2 from $x \to v_i$, there exist k out-neighbors of x that are outside N_i. Call this set N_x. Since the diameter is 2, there are k internally disjoint paths of the form $N_x \to w \to y$ where w can be anywhere. Hence, there are k internally disjoint rainbow paths from x to y of the form $x \to N_x \to w \to y$.

This completes the proof of Theorem 4. □

5 Concluding Remarks

Unfortunately our method used in the proofs of Theorems 2 and 4 does not extend to tournaments of diameter larger than 2 so the question of a sharp result in Question 1 and Problem 1 remains open even for diameter 3. The bottleneck is clearly going from vertices in A_1 to vertices in A_2 as defined in the proofs.

References

[1] Noga Alon and Joel H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster. On rainbow connection. *Electron. J. Combin.*, 15(1):Research paper 57, 13, 2008.

[3] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. Rainbow connection in graphs. *Math. Bohem.*, 133(1):85–98, 2008.

[4] L. Chen, X. Li, and H. Lian. Nordhaus-Gaddum-type theorem for rainbow connection number of graphs. *Graphs Combin.*, 29(5):1235–1247, 2013.

[5] D. Dellamonica, Jr., C. Magnant, and D. M. Martin. Rainbow paths. *Discrete Math.*, 310(4):774–781, 2010.

[6] T. P. Dorbec, I. Schiermeyer, E. Sidorowicz, and E. Sopena. Rainbow connection in oriented graphs. *Manuscript*.

[7] A. Kemnitz, J. Przybyło, I. Schiermeyer, and M. Woźniak. Rainbow connection in sparse graphs. *Discuss. Math. Graph Theory*, 33(1):181–192, 2013.

[8] X. Li, M. Liu, and I. Schiermeyer. Rainbow connection number of dense graphs. *Discuss. Math. Graph Theory*, 33(3):603–611, 2013.

[9] X. Li, Y. Shi, and Y. Sun. Rainbow connections of graphs: a survey. *Graphs Combin.*, 29(1):1–38, 2013.

[10] I. Schiermeyer. Bounds for the rainbow connection number of graphs. *Discuss. Math. Graph Theory*, 31(2):387–395, 2011.