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A Constructive Proof of the Borel-Weil Theorem for Classical Groups

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(Under the Direction of François Ziegler)

ABSTRACT

The Borel-Weil theorem is usually understood as a realization theorem for representations that have already been shown to exist by other means ("Theorem of the Highest Weight"). In this thesis we turn the tables and show that, at least in the case of the classical groups $G = U(n)$, $SO(n)$ and $Sp(2n)$, the Borel-Weil construction can be used to quite explicitly prove existence of an irreducible representation with highest weight λ, for each dominant integral form λ on the Lie algebra of a maximal torus of G.

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CHAPTER 1
INTRODUCTION

The classification of all irreducible representations of a compact connected Lie group G is given by the Theorem of the Highest Weight: *Every irreducible* G*-module* V *has a unique highest weight* λ*, which characterizes* V *and can be any so-called "dominant integral form"*. One assertion of this statement, namely the existence of the highest weight, is relatively easy. Indeed one can prove it in the $SU(2)$ case with raising/lowering operations; then once the $SU(3)$ case is understood it quickly becomes clear how to argue for general compact Lie groups. We refer the reader to B. Hall's book [**H03**, Thm 7.15] for a nice exposition of this theory, which we briefly review in Chapter 3, after a few preliminaries dealt with in Chapter 2.

The hard part of the Theorem of the Highest Weight, the second part, tells us that for every dominant integral form λ there exists an irreducible representation V_λ with highest weight λ. There are four standard approaches to constructing V_λ:

1. Cartan's case-by-case construction [**C13**]
2. Weyl's theory [**W25**]
3. Verma modules [**C48**; **H51**]
4. Borel-Weil theory [**S54**]

Historically, Cartan's approach was the earliest; he constructed irreducible representations for each compact simple Lie group (four classical series and five exceptional), considering them case-by-case. The other methods provide a more unified approach for all compact Lie groups at a time. In this exposition, we are interested in the Borel-Weil theory, but take a case-by-case viewpoint much like Cartan's.

The Borel-Weil Theorem was presented by Serre at the May 1954 Bourbaki seminar; Borel and Weil themselves never wrote it up for publication. This theorem realizes V_λ as the space of (anti-)holomorphic sections of a certain holomorphic line bundle. Namely, let G be a compact connected Lie group and λ a dominant integral form. Now λ gives us a character on the maximal torus T of G by $exp(H) \mapsto e^{i\langle\lambda,H\rangle}$, which extends to a holomorphic character χ on a Borel subgroup B of $G_{\mathbb{C}}$. We construct the line bundle over $G_{\mathbb{C}}/B$ as follows

$$\pi : G_{\mathbb{C}} \times_B \mathbb{C} \to G_{\mathbb{C}}/B. \tag{1.1}$$

Antiholomorphic sections of it are the antiholomorphic functions on $G_{\mathbb{C}}$ such that $F(gb) = \overline{\chi(b)}F(g)$ for $b \in B$ and $g \in G_{\mathbb{C}}$. We denote the space of antiholomorphic sections by $H^0(\lambda)$. The group G acts on it by $(gF)(g') = F(g^{-1}g')$ and we have the Borel-Weil Theorem: *The space* $H^0(\lambda)$ *is non-zero and forms an irreducible representation with highest weight* λ.

Traditionally this theorem has been understood as a *realization theorem* for representations that were long known to exist. From this point of view the proof is not hard either: *assuming* V_λ *exists*, we can map it into $H^0(\lambda)$ by $\varphi \mapsto F$,

$$F(g) = (gv_0, \varphi) \tag{1.2}$$

where v_0 is a highest weight vector and $(\cdot,\cdot)$ is a G-invariant inner product on V_λ. That this map is one-to-one and onto follows from a nice argument of Kobayashi and Kunze [**K61**; **K62**], who use the fact that the space $H^0(\lambda)$ has a reproducing kernel (equivalently, point evaluations on it are continuous) to show that it is automatically irreducible (or zero).

Borel and Weil also gave a second realization of V_λ as $H^0(L)$, the module of antiholomorphic sections of a line bundle $L \to X$ over a deeper quotient $X = G_{\mathbb{C}}/P$. Nowadays one recognizes X as the coadjoint orbit of λ in $\mathfrak{g}^*$, and this construction as an instance of *geometric quantization* or the *orbit method* [**K70**; **S70**; **K72**]. Chapters 4 and 5 are devoted to an exposition (with proofs) of this theory and both realizations of V_λ.

Now it is natural to ask whether the tables can be turned and the Borel-Weil construction (1.1) used to establish V_λ's very existence. In view of the Kobayashi-Kunze result, this amounts to showing that $H^0(\lambda) \neq 0$ *without assuming* V_λ *exists*. To our knowledge this was first done in [**S68**, pp. 216–217], by a method also exposed in [**H75**, 31.4] and modified in [**J87**, p. 201]. These authors set out to find a nonzero *highest weight* vector $F \in H^0(\lambda)$, and observe that this F should satisfy

$$F(\overline{n}b) = \overline{\chi(b)}F(1) \tag{1.3}$$

for all $b \in B$ and $\overline{n} \in N^-$, the unipotent radical of the opposite Borel subgroup. So apart from a constant factor, F is known on the open dense "big cell" $N^-B \subset G_{\mathbb{C}}$ of the Bruhat decomposition (see Appendix), and matters are reduced to showing that (1.3) extends antiholomorphically from N^-B to all of $G_{\mathbb{C}}$. This Steinberg does by invoking two rather deep results of Chevalley [**H75**, 5.3B and 11.2], and Jantzen by first extending (1.3) to codimension 1 Bruhat cells and then applying a version of the Riemann Extension Theorem [**G84**, p. 132].

The main observation of this Thesis, inspired by Problems of Goodman-Wallach [**G98**, 12.1.4.5] and Hall [**H03**, 7.7.6], is that in the case of the classical groups $U(n)$, $SO(n)$ and $Sp(2n)$ one can bypass this tedious and non-constructive procedure, and instead heuristically derive and then prove explicit formulas for F. Thus for instance in the case of $G = U(n)$, where λ is a diagonal matrix with integer entries $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, we find that the extension of (1.3) to $G_{\mathbb{C}} = GL(n,\mathbb{C})$—and thus a highest weight vector in $H^0(\lambda)$—is given by the formula (already known to

Cartan [**C13**, §36]!)

$$F(g) = \overline{\det_1(g)^{\lambda_1-\lambda_2}\det_2(g)^{\lambda_2-\lambda_3}\dots\det_n(g)^{\lambda_n-\lambda_{n+1}}} \tag{1.4}$$

where $\det_i(g)$ denotes the i-th principal minor of the matrix g and we have set $\lambda_{n+1} = 0$. Note that since the exponents $\lambda_i - \lambda_{i+1}$ are integers and non-negative except perhaps the last one, the function (1.4) is indeed well-defined and antiholomorphic on all of $G_{\mathbb{C}}$. Our concluding Chapter 6 is devoted to establishing this formula and similar ones for $SO(n)$ and $Sp(2n)$.

Thereby *the non-vanishing of* $H^0(\lambda)$, *the existence of* V_λ *(Theorem of the Highest Weight), and the equality of the two (Borel-Weil Theorem) are simultaneously and constructively proved for each dominant integral form* λ *on each compact classical group.*

CHAPTER 2
BASIC DEFINITIONS

In this chapter we give basic definitions and theorems.

2.1 Lie groups and Lie algebras

Definition 2.1. A *Lie group* G is a differentiable manifold which is also endowed with a group structure such that multiplication and inversion are smooth maps.
A map $\varphi : G \to H$ is a *Lie group homomorphism* if φ is both smooth and a group homomorphism. We call φ an *isomorphism* if, in addition, φ is a diffeomorphism.

Definition 2.2. A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) such that for all $X, Y, Z \in \mathfrak{g}$,

(a) $[X, Y] = -[Y, X]$ (skew-symmetry)

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

A *morphism of Lie algebras* is a linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ which preserves the bracket.

To define the Lie algebra of a Lie group we need the following:

Definition 2.3. Let $g \in G$. *Left translation* by g is the diffeomorphism $l_g : G \to G$ defined as follows:

$$l_g(g') = gg'$$

for all $g' \in G$.

Definition 2.4. A vector field X on G is called *left invariant*, if

$$dl_g \circ X = X \circ l_g$$

for each $g \in G$.

Proposition 2.5. *Let G be a Lie group and $\mathfrak{g}$ its set of left invariant vector fields.*

(a) $\mathfrak{g}$ *is a real vector space isomorphic to the tangent space $T_e(G)$ to G at the identity;*

(b) $\mathfrak{g}$ *forms a Lie algebra under the Lie bracket operation on vector fields, namely*

$$[X, Y] = X \circ Y - Y \circ X.$$

Definition 2.6. We define the *Lie algebra of the Lie group* G to be the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G. Alternatively, we could take as the Lie algebra of G the tangent space $T_e(G)$ at the identity.

Example 2.7. The group $GL(n, \mathbb{C})$ of all invertible complex matrices is a Lie group. Its Lie algebra is the vector space $\mathfrak{gl}(n, \mathbb{C})$ of all $n \times n$ complex matrices. The bracket is given by the commutator

$$[A, B] = AB - BA.$$

The Lie groups that are subgroups of $GL(n, \mathbb{C})$ are called *matrix Lie groups*.

2.2 Exponential map

The existence and uniqueness theorem for first-order differential equations gives us the next result:

Proposition 2.8. *Let G be a Lie group together with its Lie algebra $\mathfrak{g}$. Then for each $X \in \mathfrak{g}$ there exists a unique morphism of Lie groups $\exp_X : \mathbb{R} \to G$ such that $\frac{d}{dt}\exp_X(t)|_{t=0} = X$. We call this map the* one-parameter subgroup *corresponding to X.*

Definition 2.9. We define the *exponential map*

$$\exp : \mathfrak{g} \to G$$

by setting

$$\exp(X) = \exp_X(1).$$

The exponential map for matrix groups is given by ordinary exponentiation of matrices. For example, the map $\exp : \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$ sends a matrix $A \in \mathfrak{gl}(n, \mathbb{C})$ to the invertible matrix e^A, where

$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^j}{j!} + \ldots .$$

Every morphism of Lie groups $\varphi : G \to H$ defines a morphism of Lie algebras $d\varphi : \mathfrak{g} \to \mathfrak{h}$. We can construct it as follows. The derivative at the identity of the map

$$\varphi : G \to H$$

is

$$(d\varphi)_e : T_e(G) \to T_e(H)$$

which is essentially the same as

$$(d\varphi)_e : \mathfrak{g} \to \mathfrak{h}.$$

We denote this map again by $d\varphi$. It happens that $d\varphi$ preserves the bracket [**H03**, Thm 2.21], so it's actually a morphism between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

Using the exponential map we can establish the following: For $\varphi : G \to H$ and corresponding $d\varphi : \mathfrak{g} \to \mathfrak{h}$

$$\varphi(e^{tX}) = e^{t\,d\varphi(X)};$$

in particular,

$$\varphi(e^{X}) = e^{d\varphi(X)} \tag{2.1}$$

and

$$d\varphi(X) = \frac{d}{dt}\varphi(e^{tX})\bigg|_{t=0}. \tag{2.2}$$

2.3 Representations

Definition 2.10. A *representation of a Lie group* G is a complex vector space V together with a Lie group homomorphism $\Pi : G \to GL(V)$. Analogously, we define a *representation of a Lie algebra* $\mathfrak{g}$ as a complex vector space V together with a Lie algebra morphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Often we use a module notation, writing gv for $\Pi(g)v$ and Zv for $\pi(Z)v$. A *morphism* (or an *intertwining operator*) between representations V, W is a linear map $\Phi : V \to W$ such that

$$\Phi(gv) = g\Phi(v),$$

for all $g \in G$ and $v \in V$.

Definition 2.11. A subspace U of V is called *invariant* if for all $u \in U$ and $g \in G$ there holds $gu \in U$.

Definition 2.12. A representation is called *irreducible* if it is nonzero and has no non-trivial invariant subspaces.

Every finite-dimensional representation Π of G gives rise to an infinitesimal representation $\pi = d\Pi$ of $\mathfrak{g}$:

$$\Pi : G \to GL(V)$$

gives

$$\pi : \mathfrak{g} \to gl(V).$$

Using the exponential map

$$\exp : \mathfrak{g} \to G$$

the relations (2.2) and (2.1) for the representation $\Pi : G \to GL(V)$ become as follows:

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\bigg|_{t=0} \tag{2.3}$$

and

$$\Pi(e^X) = e^{\pi(X)}. \tag{2.4}$$

2.4 The adjoint representation

Let G act on itself by conjugation:

$$g \in G \mapsto c_g \quad (c_g : G \to G)$$

where c_g denotes the inner automorphism

$$c_g : g' \mapsto gg'g^{-1}.$$

Passing to the tangent spaces we obtain a Lie algebra automorphism

$$d(c_g) : \mathfrak{g} \to \mathfrak{g}.$$

Denoting by gXg^{-1} the image of X under $d(c_g)$ we get the so-called *adjoint action* of G on $\mathfrak{g}$:

$$g \mapsto Ad(g), \quad Ad(g)(X) = gXg^{-1}.$$

From Ad we can obtain infinitesimally

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

Then

$$ad(X)(Y) = [X, Y],$$

where $[X, Y]$ is the Lie bracket in $\mathfrak{g}$.

2.5 Matrix Lie groups

Let G be a matrix Lie group, i.e. $G \subset GL(n, \mathbb{C})$. Then its Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \left\{ H \in \mathfrak{gl}(n, \mathbb{C}) : e^{tH} \in G \text{ for all } t \in \mathbb{R} \right\}.$$

The bracket in this case is the commutator of two matrices X and Y, or

$$[X, Y] = XY - YX \text{ in } \mathfrak{gl}(n, \mathbb{C}).$$

And the adjoint action becames the matrix conjugation

$$Ad(g)(X) = gXg^{-1}.$$

2.6 Unitary representations

Definition 2.13. A complex representation Π of a group G is called *unitary* if there is a G-invariant inner product $\langle \cdot, \cdot \rangle$ on V:

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w \in V$. Equivalently, $\Pi(G)$ lies in the unitary group $U(V)$.

Proposition 2.14. *Every finite-dimensional unitary representation is completely reducible.*

Proof. Suppose $W \subset V$ is an invariant subspace. Then $V = W \oplus W^{\perp}$ and $W^{\perp}$ is an invariant subspace too. Indeed, for any $w' \in W^{\perp}$:

$$\langle gw', v \rangle = \langle w', g^{-1}v \rangle = 0, \text{ for all } v \in W,$$

hence $gw' \in W^{\perp}$. Repeating this procedure, we decompose V into a direct sum of irreducible invariant subspaces (this process stops because V is finite dimensional; we reduce its dimension at each step). $\square$

2.7 Representations of compact groups

Proposition 2.15. *Let G be a compact Lie group. There is a unique measure on G, the so-called left Haar measure dg, such that*

$$\int_G dg = 1$$

and which is invariant under the left action of G on itself (Definition 2.3).

The existence of Haar measure on compact Lie groups [**H63**, p. 185] gives us the next result, which is analogous to the case of finite groups:

Proposition 2.16. *Any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible.*

Proof. This argument is often called Weyl's unitarian trick. Suppose that Π is a finite-dimensional representation of a compact group G acting on a space V. We start from any inner product $\langle \cdot, \cdot \rangle$ on V and then "average" it by using the group action:

$$\langle v, w \rangle_G = \int_G \langle gv, gw \rangle dg$$

where dg is Haar measure on G. Then $\langle v, v\rangle_G > 0$ as an integral of a positive function, and by the left invariance of Haar measure

$$\langle hv, hw\rangle_G = \int_G \langle ghv, ghw\rangle dg = \langle v, w\rangle_G \tag{2.5}$$

for $h \in G$. Hence Π is a unitary representation with respect to $\langle\cdot,\cdot\rangle_G$ and thus completely reducible by Proposition 2.14. □

2.8 Schur's Lemma

Theorem 2.17 (Schur's Lemma). *Let V, W be two complex G-modules.*

(1) If V and W are irreducible, then any morphism $\Phi : V \to W$ is either zero or isomorphism.

(2) If V is irreducible and $\Phi : V \to V$ is an (auto)morphism, then $\Phi = \lambda 1$ for some $\lambda \in \mathbb{C}$.

(3) Let V and W be irreducible and let Φ_1, Φ_2 be two nonzero morphisms $V \to W$, then $\Phi_1 = \lambda\Phi_2$ for some $\lambda \in \mathbb{C}$.

Theorem 2.18. *Every irreducible representation of a compact group is finite-dimensional.*

Proof. (Cf. [**H63**, p. 344].) Let V be an irreducible G-module together with the G-invariant Haar measure dg. We fix a unit vector $v \in V$ and consider

$$\Phi = \int_G gv\langle gv, \cdot\rangle dg.$$

For $h \in G$,

$$h\Phi h^{-1} = \int_G hgv\langle gv, h^{-1}\cdot\rangle dg$$
$$= \int_G hgv\langle hgv, \cdot\rangle dg = \int_G kv\langle kv, \cdot\rangle dk = \Phi.$$

So $h\Phi = \Phi h$ i.e $\Phi : V \to V$ is an intertwining map. By Schur's lemma (2.17)

$$\Phi = \lambda 1$$

where

$$\lambda = \langle v, \Phi v\rangle = \int_G |\langle v, gv\rangle|^2 dg > 0.$$

Suppose now that E is any finite-dimensional subspace of V, together with the orthogonal projector $p : V \to E$. Then $\Phi = \lambda 1$ implies $p\Phi p = \lambda p^2 = \lambda p$, i.e.

$$\int_G pgv\langle gv, p\cdot\rangle dg = \lambda p.$$

Taking the trace of both sides gives us

$$\int_G \|pgv\|^2 dg = \lambda \dim(E).$$

At the same time, $\int_G \|pgv\|^2 dg \leqslant \int_G \|gv\|^2 dg = \mathrm{Vol}(G)$, so that

$$\dim(E) \leqslant \frac{\mathrm{Vol}(G)}{\lambda}$$

which gives an upper bound on the dimension of any E and completes the proof. $\square$

CHAPTER 3
HIGHEST WEIGHT THEORY FOR COMPACT GROUPS

Definition 3.1. Let G be a compact connected Lie group. A *maximal torus* $T \subset G$ is a maximal connected abelian subgroup of G.

Theorem 3.2. *In general, in every compact Lie group there are maximal tori and they are conjugate, i.e. if T and T' are two maximal tori then $T' = gTg^{-1}$ for some $g \in G$.*

Proof. For the existence part, take a nonzero $X \in \mathfrak{g}$, and consider $U = \overline{\exp(\mathbb{R}X)}$. Clearly U is a connected abelian subgroup of G and hence is a torus in G. Now, either U is maximal or U is contained in a larger torus. The existence of a maximal torus now follows from dimension reasons.
We prove the conjugacy part of this theorem in Chapter 4 (Theorem 4.5). □

A maximal torus T has a Lie algebra $\mathfrak{t}$, a so-called *Cartan subalgebra*, together with its dual, denoted by $\mathfrak{t}^*$.

We fix a maximal torus T in G throughout. The conjugacy property assures us that all subsequent constructions (root system, Weyl group, etc.) do not depend on the choice of the maximal torus.

3.1 Weights

Definition 3.3. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. The *complexification* $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is defined as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. We can extend the domain of π to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, in order to obtain $\pi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$.

Let G be a compact Lie group and $\Pi : G \to GL(V)$ a finite-dimensional representation of G. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and write $\mathfrak{t}_{\mathbb{C}}$ for its complexification. There exists a G-invariant inner product $(\cdot, \cdot)$ on V by Proposition 2.16. Also, $\pi = d\Pi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$ is skew-Hermitian on $\mathfrak{g}$ and Hermitian on $i\mathfrak{g}$. Hence $\mathfrak{t}_{\mathbb{C}}$ acts on V as a family of commuting normal operators and so V is simultaneously diagonalizable under the action of $\mathfrak{t}_{\mathbb{C}}$. Then we can define:

Definition 3.4. We call a form $\mu \in \mathfrak{t}^*$ a *weight* of V if

$$V^{\mu} = \{v \in V : Hv = i\langle \mu, H\rangle v \quad \forall H \in \mathfrak{t}_{\mathbb{C}}\}$$

is non-zero. This gives us a decomposition

$$V = \bigoplus_{\mu:\text{ weight}} V^{\mu},$$

which is called the *weight space decomposition* of V with respect to $\mathfrak{t}_{\mathbb{C}}$.

3.2 Roots

Consider the adjoint action $Ad : G \to GL(\mathfrak{g})$. For every $Ad(g)$ we can extend its domain from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$. Then $Ad : G \to GL(\mathfrak{g}_{\mathbb{C}})$ is a representation of G with differential ad extended by complex linearity. It has a weight space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha\text{: weight}} \mathfrak{g}^{\alpha}$$

Notice the zero weight space is $\mathfrak{g}^0 = \{Z \in \mathfrak{g}_{\mathbb{C}} : [H, Z] = 0, H \in \mathfrak{t}_{\mathbb{C}}\}$. Then $\mathfrak{g}^0 = \mathfrak{t}_{\mathbb{C}}$ since $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{g}$.

Definition 3.5. The nonzero weights of Ad are called the *roots*. Hence, we have the *root space decomposition* of $\mathfrak{g}_{\mathbb{C}}$,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\text{: root}} \mathfrak{g}^{\alpha}$$

where $\mathfrak{g}^{\alpha} = \{Z \in \mathfrak{g}_{\mathbb{C}} : [H, Z] = \alpha(H)Z, H \in \mathfrak{t}_{\mathbb{C}}\}$. We denote the set of all roots by R.

Theorem 3.6. *Let* $\Pi : G \to GL(V)$ *be a representation of* G*, and* $\mathfrak{t}$ *a Cartan subalgebra of* $\mathfrak{g}$*. For any root* α *and weight* μ

1. $\pi(\mathfrak{g}^{\alpha})V^{\mu} \subset V^{\mu+\alpha}$
2. *in particular, for roots* α *and* β*,* $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}$.

Definition 3.7. A form $\mu \in \mathfrak{t}^*$ is called *integral* if the character $e^{i\langle \mu, \cdot \rangle}$ of $\mathfrak{t}$ factors through the covering $exp : \mathfrak{t} \to T$.

3.3 Weyl group

Definition 3.8. The *Weyl group* of G is $N(T)/T$. Here T is the maximal torus and $N(T) = \{n \in G : nTn^{-1} \subset T\}$ is its normalizer.

Up to isomorphism, the Weyl group is independent of the choice of maximal torus. Given $w \in N(T), H \in \mathfrak{t}$, and $\lambda \in \mathfrak{t}^*$, define an action of $N(T)$ on $\mathfrak{t}$ and $\mathfrak{t}^*$ by

$$w(H) = Ad(w)(H),$$
$$\langle w(\lambda), H\rangle = \langle \lambda, Ad(w^{-1})(H)\rangle.$$

As $Ad(T)$ acts trivially on $\mathfrak{t}$, the action of $N(T)$ descends to an action of $W = N(T)/T$.

Theorem 3.9 (Elementary properties of the action of the Weyl group).

1. *W is a (compact and discrete, hence) finite group.*
2. *The set of roots R is W-invariant.*
3. $\mathrm{Ad}(w)(\mathfrak{g}^\alpha) = \mathfrak{g}^{w(\alpha)}$.

Definition 3.10. A *Weyl chamber* $C \subset \mathfrak{t}$ is a chosen connected component of

$$\mathfrak{t} \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha).$$

Definition 3.11. The *system of positive roots* R^+ associated to C is the set

$$R^+ = \{\alpha \in R : \langle \alpha, H \rangle > 0 \text{ for all } H \in C\}.$$

In the same way, we define the *system of negative roots*

$$R^- = \{\alpha \in R : \langle \alpha, H \rangle < 0 \text{ for all } H \in C\}.$$

We write

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}^\alpha,$$

so that we have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+$$

by the root space decomposition. This is called the triangular decomposition, in analogy with the decomposition of $\mathfrak{gl}(n, \mathbb{C})$ into diagonal and strictly upper/lower triangular matrices.

3.4 Order on $\mathfrak{t}^*$

We can define a *partial order* $\leqslant$ on $\mathfrak{t}^*$ as follows:

$$\lambda \leqslant \mu \text{ iff } \mu - \lambda \text{ is non-negative on } \bar{C}.$$

Definition 3.12. The *dominant chamber* $D \subset \mathfrak{t}^*$ consists of those $\mu \in \mathfrak{t}^*$ such that $w(\mu) \leqslant \mu$ for all $w \in W$.

3.5 Highest weight

Now, as explained in [**B05**, §IX.7], $\bar{C}$ and D are fundamental domains for the action of W on $\mathfrak{t}$ and $\mathfrak{t}^*$, in the sense that each orbit intersect $\bar{C}$ and D in a single point, and we have:

Theorem 3.13 (Half the Theorem of the Highest Weight). *Every irreducible* G*-module* V *has a unique* $\leqslant$*-maximal weight* λ*, which characterizes* V*. The highest weight* λ *is an integral point of* D*, and we have* $\dim(V^\lambda) = 1$*.*

As explained in the Introduction, the harder step (the other half) is to prove the converse of Theorem 3.13: *Every dominant integral form is the highest weight of some irreducible representation.* In what follows we will show how to construct an irreducible representation with given highest weight by the Borel-Weil method.

CHAPTER 4
COADJOINT ORBITS

4.1 The coadjoint action

We have the natural adjoint action (§2.4) of the Lie group G on its Lie algebra $\mathfrak{g}$:

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}.$$

This gives rise to the *coadjoint action* of G on the dual space $\mathfrak{g}^*$ as follows:

$$\mathrm{coAd}(g) : \mathfrak{g}^* \to \mathfrak{g}^*$$

given by

$$\langle \mathrm{coAd}(g)(x), Z \rangle = \langle x, \mathrm{Ad}(g^{-1})(Z) \rangle$$

for $x \in \mathfrak{g}^*$, $g \in G$, $Z \in \mathfrak{g}$. Infinitesimally,

$$\mathrm{coad}(X) : \mathfrak{g}^* \to \mathfrak{g}^*$$

is given by

$$\langle \mathrm{coad}(X)(y), Z \rangle = \langle y, -\mathrm{ad}(X)(Z) \rangle = \langle y, [Z, X] \rangle,$$

where $X, Z \in \mathfrak{g}$ and $y \in \mathfrak{g}^*$.

Theorem 4.1. *Let G be a compact Lie group, $T \subset G$ a maximal torus, $\mathfrak{t}$ its Lie algebra. Then $\mathfrak{t} = \{T\text{-fixed points in } \mathfrak{g}\}$.*

Proof. "$\subset$" part. Let $H \in \mathfrak{t}$, then $e^{sH} \in T$ for all $s \in \mathbb{R}$. Then

$$\mathrm{Ad}(t)(H) = \frac{d}{ds}(te^{sH}t^{-1})|_{s=0} = \frac{d}{ds}(e^{sH})|_{s=0} = H,$$

since t and e^{sH} are elements in T and thus commute.

"$\supset$" part. Take $Z \in \mathfrak{g}$. Suppose that $\mathrm{Ad}(t)(Z) = Z$ for all $t \in T$. As we know,

$$e^{\mathrm{Ad}(t)(sZ)} = te^{sZ}t^{-1}, \quad \forall s \in \mathbb{R}$$

which in our case becomes

$$e^{sZ} = te^{sZ}t^{-1}, \quad \forall t \in T.$$

Thus e^{sZ} commutes with all elements in T. So T and $\{e^{sZ}\}$ generate a larger torus. But T is maximal, hence $e^{sZ} \in T$, for all $s \in \mathbb{R}$, and therefore $Z \in \mathfrak{t}$. $\square$

Hence we have a natural projection

$$\Pi = \int_T \mathrm{Ad}(t)dt : \mathfrak{g} \to \mathfrak{t},$$

which is onto. Indeed, for all $t \in T$: $\mathrm{Ad}(t)(\Pi(Z)) = \mathrm{Ad}(t)\int_T \mathrm{Ad}(s)(Z)ds = \int_T \mathrm{Ad}(ts)(Z)ds = \int_T \mathrm{Ad}(r)(Z)d(t^{-1}r) = \Pi(Z)$.

Therefore we have a natural injection $\Pi^* : \mathfrak{t}^* \to \mathfrak{g}^*$. (We define a transpose map $\langle \Pi^*(\lambda), Z\rangle := \langle \lambda, \Pi(Z)\rangle$, where $Z \in \mathfrak{g}$, $\Pi(Z) \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}^*$.)

Lemma 4.2. *Π^* identifies $\mathfrak{t}^*$ with the set of all T-fixed points in $\mathfrak{g}^*$.*

Proof. "$\subset$": We want to show that $\Pi^*(\mathfrak{t}^*)$ consists of T-fixed points.

Let $x = \Pi^*(\lambda)$ and $s \in T$, we want to show $s(x) = x$. But for $Z \in \mathfrak{g}$:

$$\begin{aligned} \langle s(x), Z\rangle &= \langle x, \mathrm{Ad}(s^{-1}(Z))\rangle \\ &= \langle \lambda, \Pi(\mathrm{Ad}(s^{-1}(Z)))\rangle \\ &= \langle \lambda, \int_T \mathrm{Ad}(ts^{-1})(Z)dt\rangle \\ \text{by invariance of } dt \quad &= \langle \lambda, \Pi(Z)\rangle = \langle x, Z\rangle \end{aligned}$$

"$\supset$": We want to show that every T-fixed point in $\mathfrak{g}^*$ belongs to $\Pi^*(\mathfrak{t}^*)$.

Let $x \in \mathfrak{g}^*$ be T-fixed, put $\lambda = x|_{\mathfrak{t}}$. Then, for $Z \in \mathfrak{g}$ hence for $\Pi(Z) \in \mathfrak{t}$ we have:

$$\langle \Pi^*(\lambda), Z\rangle = \langle \lambda, \Pi(Z)\rangle = \langle x, \Pi(Z)\rangle$$
$$= \langle x, \int_T \mathrm{Ad}(t)(Z)dt\rangle = \langle \int_T t^{-1}(x)dt, Z\rangle.$$

But x is T-fixed, hence $\langle \Pi^*(\lambda), Z\rangle = \langle x, Z\rangle$. □

Definition 4.3. Given $x \in \mathfrak{g}^*$, the *coadjoint orbit* $X = G(x)$ is the image of the map

$$g \in G \mapsto \mathrm{coAd}(g)(x) \in \mathfrak{g}^*.$$

Such an orbit has the form G/G_x, where G_x is stabilizer of $x \in \mathfrak{g}^*$ under the coadjoint action.

4.2 Hunt's argument and its consequences

Theorem 4.4. *Every coadjoint orbit X of G meets $\mathfrak{t}^*$.*

Proof. (Cf. [**H56**; **B79**].) Let $A \in \mathfrak{t}$ generate a dense 1-parameter subgroup in T: $\overline{\exp(\mathbb{R}A)} = \overline{\{e^{sA} : s \in \mathbb{R}\}} = T$. And let $x \in X$ be a point where the "hamiltonian" $H_A(x) := \langle x, A\rangle$ attains its minimum (exists by compactness of the orbit as image of a compact group). Then $DH_A(x) = 0$, $DH_A(x) : T_xX \to \mathbb{R}$. Evaluating $DH_A(x)$ on the tangent vectors[1] $Z(x) = \frac{d}{dt}e^{tZ}(x)|_{t=0}$ we get

$$0 = DH_A(x)(Z(x)) = \langle Z(x), A\rangle$$

In our case $x \in \mathfrak{g}^*$ and $\langle Z(x), Z'\rangle = \langle x, [Z', Z]\rangle$. Then,

$$\langle Z(x), A\rangle = -\langle x, [Z, A]\rangle = -\langle A(x), Z\rangle.$$

As this vanishes for all $Z \in \mathfrak{g}$, we conclude that $A(x) = 0$ and hence

$$t(x) = x \quad \forall t \in \overline{exp(\mathbb{R}A)} = T.$$

Hence, x is a fixed point and $x \in \mathfrak{t}^*$= {T-fixed points in $\mathfrak{g}^*$}. □

Theorem 4.5 (Corollary). *Any two maximal tori are conjugate.*

Proof. Let $T = \overline{\exp(\mathbb{R}A)}$ and $U = \overline{\exp(\mathbb{R}B)}$ be two maximal tori. We can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by a G-invariant inner product on $\mathfrak{g}$. By the theorem above there is a $g \in G$ such that $Ad(g)(B) \in \mathfrak{t}$. Thus $g exp(tB)g^{-1} = exp(tAd(g)(B)) \subset T \quad \forall t \in \mathbb{R}$. Hence $gUg^{-1} \subset T$ by continuity. Hence $gUg^{-1} = T$ by maximality. □

Theorem 4.6 (Corollary). *The intersection* $G(\lambda) \cap \mathfrak{t}^* = W(\lambda)$ *(is an orbit of the Weyl group).*

Proof. "$\supset$": Consider the Weyl group $W = N(T)/T$. Take $w = nT \in N(T)/T$. Then $w(\lambda) = (nT)(\lambda) = n(\lambda) \in G(\lambda)$.

"$\subset$": Suppose $g(\lambda) \in \mathfrak{t}^*$, then $g(\lambda)$ is T-fixed

$$t(g(\lambda)) = g(\lambda), \quad \forall t \in T.$$

Hence $g^{-1}Tg$ lies in the stabilizer G_λ of λ. Now, T and gTg^{-1} are two maximal tori in G_λ. So they are conjugate; let $h \in G_\lambda$ be such that

$$T = h^{-1}g^{-1}Tgh.$$

So gh belongs to the torus's normalizer, $N(T)$. Let w be the corresponding class in the Weyl group: $w = ghT$. The action of w on $\lambda \in \mathfrak{t}^*$ gives us

$$w(\lambda) = gh(\lambda) = g(\lambda),$$

hence $g(\lambda) \in W(\lambda)$ which completes the proof. □

[1]Given $x \in \mathfrak{g}^*$, the vectors $Z(x)$ with $Z \in \mathfrak{g}$, constitute the tangent space to the coadjoint orbit at the point x. See Section 4.3 on symplectic manifolds for details.

Theorem 4.7 (Corollary). *The (co)adjoint stabilizer* $G_B = \{g \in G : gBg^{-1} = B\}$ *of any* $B \in \mathfrak{g}$ *(or* $\mathfrak{g}^*$*) is connected.*

Proof. Let $a \in G_B$. Then $\overline{\exp(\mathbb{R}B)}$ is a torus contained in the centralizer $Z_G(a) = \{g \in G : gag^{-1} = a\}$, hence contained in a maximal torus U of $Z_G(a)$. Claim: $a \in U \subset G_B^0$, the connected component of the identity in G_B.

Proof of $U \subset G_B^0$: We need only show $U \subset G_B$ since U is connected. But that's clear since elements of U commute with $\exp(\mathbb{R}B)$ and hence with B.

Proof of $a \in U$: Let $a = \exp(A)$ (such an A exists by surjectiveness of the exponential map [**B05**, Cor. IX.2.2.1]), then $\overline{\exp(\mathbb{R}A)}$ is another torus in $Z_G(a) \Rightarrow$ it is conjugate by Theorem 4.5. $\exists g \in Z_G(a)$ such that $g\exp(\mathbb{R}A)g^{-1} \subset U$. In particular U contains $g\exp(A)g^{-1}=gag^{-1}=a$. □

Theorem 4.8 (Corollary). *(Co)adjoint orbits of a compact connected Lie group are simply connected.*

Proof. Consider the diagram

$$\underset{\substack{\text{closed}\\\text{subgroup}}}{H} \to \underset{\substack{\text{Lie}\\\text{group}}}{G} \to G/H.$$

It gives rise to a long exact sequence of homotopy groups, including the following piece [**S51**, §17]:

$$\cdots \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to \dots \tag{4.1}$$

Let H be a stabilizer of the (co)adjoint action. Without loss of generality we can replace G with the universal cover $\tilde{G}$, since both G and $\tilde{G}$ have the same (co)adjoint orbits. Then the above (4.1) becomes

$$\cdots \to 0 \to \pi_1(G/H) \to 0 \to 0 \to \pi_0(G/H) \to \dots$$

and thus forces $\pi_1(G/H)$ to be trivial. □

4.3 The coadjoint orbits as symplectic manifolds

Definition 4.9. A symplectic manifold is a smooth manifold X with a nondegenerate, closed 2-form σ.

Definition 4.10. A *smooth action* of G on a manifold X is a group morphism $\rho : G \to \mathrm{Diff}(X)$ of G into the diffeomorphisms of X, such that $(g, x) \mapsto \rho(g)(x)$ is a smooth map $G \times X \to X$.

It is customary to drop ρ from the notation and write $g(x)$ for $\rho(g)(x)$. The *orbit* of x is

$$G(x) = \{g(x) : g \in G\}.$$

Every orbit of a smooth action of G on a manifold X is diffeomorphic to the quotient manifold G/G_x, where $G_x = \{g \in G : g(x) = x\}$ is the *stabilizer* of x. If there is just one orbit, the action is called *transitive* and X is called a *homogeneous space* of G.

Every smooth action ρ of G on a manifold X induces an *infinitesimal action* of the Lie algebra $\rho_* : \mathfrak{g} \to Vect(X)$ by $\rho_* : Z \mapsto (d\rho)_e(Z)$, or, using the exponential map

$$\rho_*(Z)(x) = \frac{d}{dt} e^{tZ}(x)\bigg|_{t=0}.$$

As with ρ one usually drops ρ_* from the notation and writes $Z(x)$ instead of $\rho_*(Z)(x)$. If the action is transitive, the diffeomorphism $G/G_x \to G(x)$ ensures that the tangent space to an orbit $G(x)$ at x is

$$T_x G(x) = \{Z(x) : Z \in \mathfrak{g}\} = \mathfrak{g}/\mathfrak{g}_x$$

where $\mathfrak{g}_x$ is the Lie algebra of the stabilizer G_x.

Let X be a coadjoint orbit of G acting on $\mathfrak{g}^*$. Then X admits a G-invariant symplectic structure given by the Kirillov-Kostant-Souriau 2-form:

Theorem 4.11 (The Kirillov-Kostant-Souriau Theorem [**K72**; **K70**; **S70**]).

1. *Every coadjoint orbit of a Lie group is a homogeneous symplectic manifold when endowed with the KKS 2-form*

$$\sigma(Z(x), Z'(x)) = \langle x, [Z', Z] \rangle.$$

2. *Conversely, every homogeneous symplectic manifold of a connected Lie group* G *is, up to a possible covering, a coadjoint orbit of some central extension of* G.

4.4 An example of coadjoint orbits: Grassmannians

In this section we are going to illustrate the notion of coadjoint orbits and exhibit the Kirillov-Kostant-Souriau 2-form on them in the special case of Grassmannians (and projective spaces in particular).

Let V be a finite dimensional Hilbert space, so it comes equipped with a complex inner product. Based on this product we consider the group of all unitary operators, $G = U(V)$, together with its Lie algebra $\mathfrak{g} = \mathfrak{u}(V)$:

$$G = U(V) = \{g \in GL(V) : \bar{g}g = 1\},$$
$$\mathfrak{g} = \{Z \in \mathfrak{gl}(V) : \bar{Z} + Z = 0\}.$$

Here and in what follows the bar $\bar{}$ denotes adjoint. Thus $\mathfrak{g}$ consists of skew-adjoint operators. We identify $\mathfrak{g}^*$ with the space of all self-adjoint operators as follows:

$$\begin{aligned} i\mathfrak{g} &\to \mathfrak{g}^* \\ x &\mapsto \frac{1}{i}\mathrm{Tr}(x\,\cdot), \end{aligned} \tag{4.2}$$

so that $\mathfrak{g}^*$ becomes

$$\mathfrak{g}^* = \{x \in \mathfrak{gl}(V) : \bar{x} = x\}.$$

Now we restrict our attention to the case $V = \mathbb{C}^n$ in order to be more explicit, however the case of an arbitrary space V can be treated in the same way. Then G becomes $U(n)$ and the coadjoint orbits are conjugacy classes of self-adjoint matrices: $G(x) = \{gxg^{-1} \text{ for all } g \in G\}$. Each orbit intersects the dominant chamber

$$D = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_1 \geqslant \cdots \geqslant \lambda_n \right\}$$

in exactly one point (compare §3.5 and Theorem 4.6). The orbit is integral (i.e. it goes through an integral point), when all λ_i are integers.

Consider the remarkable orbits

$$\begin{aligned} X_k &= \left\{x \in \mathfrak{g}^* : x^2 = x, \mathrm{Tr}(x) = k\right\} \\ &= \{\text{rank } k \text{ self-adjoint projectors } x : V \to V\} \\ &= G\begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}, \end{aligned}$$

where n is the complex dimension of V. Note that X_1 is just $\mathbb{P}(V)$. Indeed, the map $V \setminus \{0\} \to X_1$, defined by

$$\xi \mapsto \frac{\xi(\xi, \cdot)}{\|\xi\|^2},$$

(where $(\ ,\ )$ is the inner product on V), induces a bijection between $\mathbb{P}(V)$ and X_1. Similarly, X_k is nothing but the Grassmannian of k-dimensional subspaces of V.

Proposition 4.12. *The 2-form on X_k is given by*

$$\sigma(\delta x, \delta' x) = \frac{1}{i}\mathrm{Tr}(x[\delta x, \delta' x]) = \mathrm{Tr}(\delta' x J \delta x),$$

where $\delta x, \delta' x \in T_x X_k$ and $J\delta x = \dfrac{1}{i}[x, \delta x]$ is a complex structure on X_k $(J^2 = -\mathrm{id})$.

Proof. Recall that the Kirillov-Kostant-Souriau 2-form (Theorem 4.11) is given by $\sigma(Z(x), Z'(x)) = \langle x, [Z', Z]\rangle$. First, given δx we want to find Z such that $Z(x) = \delta x$. Consider the pairing

$$\begin{aligned}\langle Z(x), Z'\rangle &= \langle x, [Z', Z]\rangle = \frac{1}{i}\mathrm{Tr}(x[Z', Z]) = \frac{1}{i}\mathrm{Tr}(xZZ' - xZZ') \\ &= \frac{1}{i}\mathrm{Tr}((Zx - xZ)Z') = \frac{1}{i}\mathrm{Tr}([Z, x]Z') = \langle [Z, x], Z'\rangle.\end{aligned}$$

Thus $Z(x) = [Z, x]$. Our claim is that given δx, Z can be taken equal to $[\delta x, x]$. Indeed, $x^2 = x$ implies $x\delta x x = 0$, and $Z(x)$ becomes

$$\begin{aligned}Z(x) &= [Z, x] = [[\delta x, x], x] = [\delta x x - x\delta x, x] \\ &= \delta x x - x\delta x x - x\delta x x + x\delta x = \delta x x + x\delta x = \delta[x^2] = \delta x.\end{aligned}$$

Therefore we can compute

$$\begin{aligned}\sigma(\delta x, \delta' x) = \langle x, [Z', Z]\rangle = \langle Z(x), Z'\rangle &= \langle \delta x, [\delta' x, x]\rangle = \frac{1}{i}\mathrm{Tr}(\delta x[\delta' x, x]) \\ &= \frac{1}{i}\mathrm{Tr}(\delta x\delta' x x - \delta x x\delta' x) = \frac{1}{i}\mathrm{Tr}(x[\delta x, \delta' x]).\end{aligned}$$

Also defining a linear operator J on each tangent space by $J\delta x = \frac{1}{i}[x, \delta x]$ we get $J^2\delta x = -[x, [x, \delta x]] = -\delta x$ so that J is an almost complex structure on X_k. That gives another expression for the 2-form σ:

$$\sigma(\delta x, \delta' x) = \frac{1}{i}\mathrm{Tr}(\delta' x x\delta x - \delta' x\delta x x) = \mathrm{Tr}(\delta' x\frac{1}{i}[x, \delta x]) = \mathrm{Tr}(\delta' x J\delta x). \qquad \square$$

Together σ and J define a pseudo Riemannian metric $g(\cdot, \cdot)$ on X_k

$$g(\delta x, \delta' x) = \sigma(J\delta x, \delta' x) = -\mathrm{Tr}(\delta x\delta' x)$$

which is negative definite: $g(\delta x, \delta x) = -\mathrm{Tr}(\delta x^2) < 0$. Each two structures of (σ, J, g) determine the third. In that way we obtain a *Kähler structure* on X_k.

CHAPTER 5
NON-CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM

In this chapter we consider the coadjoint orbit X that goes through the highest weight λ of an irreducible representation V. Then we relate V to the symplectic G-manifold X as follows (cf. [**Z96**, pp. 47–50]):

First, we embed X into the projective space $\mathbb{P}(V)$, which gives us a hermitian line bundle with connection $L \to X$ with base X. The curvature of the connection of L equals the orbit's 2-form. (This is called prequantization.)

Second, the square integrable sections for Liouville measure on X make a unitary G-module. We extract the submodule of antiholomorphic sections $H^0(L)$ from it. (This is called polarization.)

Now in Theorem 5.4 we show that $H^0(L)$ is isomorphic to V; and in Section 5.5 we show that both are also isomorphic to the representation $H^0(\lambda)$ discussed in the Introduction.

5.1 The tautological line bundle over projective space

In this section we return our attention to the unitary G-module V. Adopting the constructions from the previous section, we consider the projective space $\mathbb{P}(V)$ as the space X_1 of all rank one self-adjoint projectors x in V:

$$\mathbb{P}(V) = \left\{ x \in \mathfrak{u}(V)^* : x^2 = x, \mathrm{Tr}(x) = 1 \right\}.$$

The *tautological line bundle* over $\mathbb{P}(V)$ is

$$L = \{(x, \xi) \in \mathbb{P}(V) \times V : \xi \in \mathrm{Im}(x)\}$$

together with the projection $p : L \to \mathbb{P}(V)$

$$p : (x, \xi) \mapsto x.$$

As we can see the fiber above x is just the complex line $\mathrm{Im}(x) = \mathbb{C}\xi$. Note that every $\xi \neq 0$ in V determines a projector $x = \dfrac{\xi\bar{\xi}}{\|\xi\|^2}$, where $\bar{\xi} = (\xi, \cdot)$. So we can identify $L^\times = L \setminus \{(x, 0) : x \in \mathbb{P}(V)\}$ with $V^\times = V \setminus \{0\}$ using the map $(x, \xi) \mapsto \xi$. We continue writing p for the projection $L^\times = V^\times \to \mathbb{P}(V)$.

Infinitesimally one can write

$$T_{(x,\xi)}L^\times = T_\xi V^\times = \xi^\perp \oplus \mathbb{C}\xi = \mathrm{Ker}(x) \oplus \mathrm{Im}(x).$$

We write $\mathrm{Hor} : T_\xi V^\times \to \mathrm{Ker}(x)$ and $\mathrm{Vert} : T_\xi V^\times \to \mathrm{Im}(x)$ for the projections associated to this decomposition (i.e. $\delta\xi = (1-x)\delta\xi + x\delta\xi$, where $x = \dfrac{\xi\bar{\xi}}{\|\xi\|^2}$). This splitting is called a *connection* on the line bundle L.

By definition ϖ is the *connection* 1-*form* on $L^\times$ such that

$$\mathrm{Vert}(\delta\xi) = i\xi\varpi(\delta\xi).$$

Explicitly,

$$x\delta\xi = i\xi\varpi(\delta\xi),$$

hence

$$\varpi(\delta\xi) = \frac{(\xi, \delta\xi)}{i\|\xi\|^2} \quad \text{is a 1-form on } L^\times.$$

Theorem 5.1. *This connection has curvature σ.*

Note: Given a connection ϖ on $L \xrightarrow{p} \mathbb{P}(V)$ its *curvature* is the 2-form ω such that

$$d\varpi = p^*\omega.$$

Proof. We compute

$$\begin{aligned}
[d\varpi](\delta\xi, \delta'\xi) &= [\delta\varpi](\delta'\xi) - [\delta'\varpi](\delta\xi) \\
&= \frac{1}{i}\left\{(\delta\frac{\xi}{\|\xi\|^2}, \delta'\xi) - (\delta'\frac{\xi}{\|\xi\|^2}, \delta\xi)\right\} \\
&= \frac{1}{i}\left\{(\frac{\delta\xi\|\xi\|^2 - \xi[(\delta\xi, \xi) + (\xi, \delta\xi))]}{\|\xi\|^4}, \delta'\xi) - (\frac{\delta'\xi\|\xi\|^2 - \xi[(\delta'\xi, \xi) + (\xi, \delta'\xi))]}{\|\xi\|^4}, \delta\xi)\right\} \\
&= \frac{1}{i}\left\{\frac{(\delta\xi, (1-x)\delta'\xi) - (\delta'\xi, (1-x)\delta\xi)}{\|\xi\|^2}\right\}
\end{aligned}$$

Now observe that[1] $x\xi = \xi$ gives $\delta x\xi + x\delta\xi = \delta\xi$, or $\delta x\xi = (1-x)\delta\xi$. Also $(1-x)^2 = (1-x)$. Hence

$$[d\varpi](\delta\xi, \delta'\xi) = \frac{1}{i}\frac{(\delta x\xi, \delta' x\xi) - (\delta' x\xi, \delta x\xi)}{\|\xi\|^2}.$$

In general, $(u, v) = \mathrm{Tr}(v\bar{u})$ gives

$$\begin{aligned}
[d\varpi](\delta\xi, \delta'\xi) &= \frac{1}{i}\left\{\mathrm{Tr}(\delta' x\frac{\xi\bar{\xi}}{\|\xi\|^2}\delta x) - \mathrm{Tr}(\delta x\frac{\xi\bar{\xi}}{\|\xi\|^2}\delta' x)\right\} \\
&= \frac{1}{i}\mathrm{Tr}(\delta' x x\delta x - \delta x x\delta' x) \\
&= \frac{1}{i}\mathrm{Tr}(\delta' x[x, \delta x]) \\
&= \mathrm{Tr}(\delta' x J\delta x) \\
&= \sigma(\delta x, \delta' x). \qquad \square
\end{aligned}$$

[1] $x\xi$ is a product of square matrix and column vector; δx and $\delta\xi$ are the derivatives (tangent vectors) in Frechet notation.

Note that every *complex* submanifold W of $\mathbb{P}(V)$ is symplectic for the restriction of σ:

(a) The 2-form on the manifold W is $i^*\sigma$, where $i : W \to \mathbb{P}(V)$ is the inclusion map. Now $d[i^*\sigma] = i^* d\sigma = 0$, thus $i^*\sigma$ on W is closed.

(b) This 2-form is non-degenerate. Indeed, $\sigma(J\delta w, \delta w) = g(\delta w, \delta w) < 0$ unless $\delta w = 0$.

5.2 The orbit of the highest weight vector

We saw that $\mathbb{P}(\mathbb{C}^n)$ is a coadjoint orbit of $U(n)$. In analogy with it, we consider $\mathbb{P}(V)$ as a coadjoint orbit of $U(V)$.

Now we return to our original notation: G is any compact connected Lie group and $\Pi : G \to U(V)$ is an irreducible representation of it. Let λ be its highest weight (so that $V = V_\lambda$) and let v_0 be a highest weight vector and $w_0 = \dfrac{v_0 \bar{v}_0}{\|v_0\|^2}$ be the corresponding point in $\mathbb{P}(V)$.

Theorem 5.2. *The orbit* $W := G(w_0)$ *is a complex (hence symplectic) submanifold of* $\mathbb{P}(V)$.

Proof. We can replace G and $\mathfrak{g}$ by their images in $\mathrm{End}(V)$. For $V = V_\lambda$ we have

$$V = \bigoplus_{\mu:\ \text{weight}} V^\mu,$$

where

$$V^\mu = \{v \in V : Hv = i\langle \mu, H\rangle v \quad \forall H \in \mathfrak{t}\}.$$

Also

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}^\alpha,$$

where

$$\mathfrak{g}^\alpha = \{Z \in \mathfrak{g}_{\mathbb{C}} : [H, Z] = i\langle \alpha, H\rangle Z \quad \forall H \in \mathfrak{t}\}.$$

We have $\mathfrak{g}^\alpha V^\mu \subset V^{\mu+\alpha}$ by Theorem 3.6. Each $\alpha \in R$ is either > 0, or < 0 (sign in the chamber C).

Claim:

1. if $\alpha > 0$ then $\mathfrak{g}^\alpha(w_0) = 0$,

2. $\mathfrak{g}^0(w_0) = 0$,

3. if $\alpha < 0$, then $\mathfrak{g}^\alpha(w_0) \subset \mathfrak{g}(w_0)$ where $\mathfrak{g}$ is the (real!) Lie algebra of G.

From this it follows that $\mathfrak{g}_{\mathbb{C}}(w_0) = \mathfrak{g}(w_0)$ and by equivariance that $\mathfrak{g}_{\mathbb{C}}(g(w_0)) = \mathfrak{g}(g(w_0))\forall g \in G$. Indeed, suppose $Z \in \mathfrak{g}_{\mathbb{C}}$ and $g \in G$. Then

$$Z(g(w_0)) = \frac{d}{dt}e^{tZ}(g(w_0))|_{t=0},$$

here the group action is $g(w_0) = gw_0g^{-1}$ (composition of operators $g \in U(V)$). Now

$$\begin{aligned}
Z(g(w_0)) &= \frac{d}{dt}gg^{-1}e^{tZ}(g(w_0))|_{t=0} \\
&= \frac{d}{dt}g(e^{tg^{-1}Zg}(w_0))|_{t=0} \qquad (= \frac{d}{dt}g(e^{tAd(g^{-1}Z)})|_{t=0}) \\
&= Dg(w_0)(Ad(g^{-1})(z)(w)) \in \mathfrak{g}(g(w_0)).
\end{aligned}$$

This shows that $\mathfrak{g}_{\mathbb{C}}(g(w_0)) \subset \mathfrak{g}(g(w_0))$ with trivial reverse inclusion.

Thus we have $\mathfrak{g}_{\mathbb{C}}(w) \subset \mathfrak{g}(w)$ for all $w \in W = G(w_0)$. It follows that $G(w_0)$ is open in $G_{\mathbb{C}}(w_0)$. Also $G(w_0)$ is compact, hence closed in $G_{\mathbb{C}}(w_0)$. Since $G_{\mathbb{C}}(w_0)$ is connected, we conclude that $W = G(w_0) = G_{\mathbb{C}}(w_0)$. So W is a complex orbit, hence a complex manifold.

It remains to show our 3 claims:

1) $\alpha > 0 \Rightarrow \mathfrak{g}^{\alpha}(w_0) = 0$.

The action on $w_0 \in \mathbb{P}(V)$ is defined by $Z(w_0) = p_*(Zv_0)$ where p_* is the derivative of $p : V \setminus \{0\} \to \mathbb{P}(V)$. But for $Z \subset \mathfrak{g}^{\alpha}$, $\alpha > 0$:

$$Zv_0 \in V^{\lambda+\alpha} = \{0\}$$

since λ is the *highest* weight. Thus $Zv_0 = 0$.

2) $\mathfrak{g}^0(w_0) = 0$

Indeed, for $Z \in \mathfrak{g}^0 = \mathfrak{t}_{\mathbb{C}}$ we have

$$\begin{aligned}
Z(w_0) &= p_*(Zv_0) \\
&= p_*(i\langle\lambda, Z\rangle v_0) \\
&= \frac{d}{dt}p(e^{it\langle\lambda,Z\rangle}v_0)|_{t=0} \\
&= \frac{d}{dt}p(v_0)|_{t=0} = 0.
\end{aligned}$$

3) If $\alpha < 0$, then $\mathfrak{g}^{\alpha}(w_0) \subset \mathfrak{g}(w_0)$.

Suppose $Z \in \mathfrak{g}^{\alpha}$. Claim: $\bar{Z} \in \mathfrak{g}^{-\alpha}$ (adjoint of $Z \in End(V)$).

Indeed, for $H \in \mathfrak{t}$ we have

$$[H, \bar{Z}] = H\bar{Z} - \bar{Z}H = -\bar{H}\bar{Z} + \bar{Z}\bar{H} = \overline{HZ - ZH} = \overline{i\langle\alpha, H\rangle Z} = -i\langle\alpha, H\rangle\bar{Z}.$$

In particular $\bar{Z}(w_0) = 0$ hence $Z(w_0) = (Z - \bar{Z})(w_0) \subset \mathfrak{g}(w_0)$. ($(Z - \bar{Z})$ is skew-adjoint, hence in $\mathfrak{g}$.) □

5.3 The moment map and coadjoint orbits

In the previous section we showed that the orbit W of the projectivized highest weight vector $w_0 \in \mathbb{P}(V)$ is a homogeneous symplectic manifold. By the Kirillov-Kostant-Souriau Theorem 4.11, the resulting moment map $\Phi : W \to \mathfrak{g}^*$ will make $W = G(w_0)$ (a covering of) a coadjoint orbit of G. In this section we verify that this orbit is exactly $G(\lambda)$.

To this end we observe that G preserves ϖ: indeed

$$(g^*\varpi)(\delta\xi) = \frac{(g\xi, g\delta\xi)}{i\|g\xi\|^2} = \frac{(\xi, \delta\xi)}{i\|\xi\|^2} = \varpi(\delta\xi).$$

It follows that the Lie derivative $L(Z)\varpi$ vanishes for all $Z \in \mathfrak{g}$. By Cartan's formula for the Lie derivative, $L(Z)\varpi = i(Z)d\varpi + di(Z)\varpi$, it follows that

$$i(Z)d\varpi = -d\langle\Phi(\cdot), Z\rangle$$

where $\langle\Phi(\cdot), Z\rangle = i(Z)\varpi$, or $\langle\Phi(\xi), Z\rangle = \varpi(Z\xi)$. By definition of a moment map [**S70**], this means that

$$\langle\Phi(\xi), Z\rangle = \varpi(\pi(Z)\xi) = \frac{(\xi, \pi(Z)\xi)}{i\|\xi\|^2}.$$

defines a moment map Φ for the action of G on $(L^\times, d\varpi)$. We can write it equivalently as

$$\begin{aligned}
\langle\Phi(\xi), Z\rangle &= \frac{1}{i}\mathrm{Tr}\left(\frac{\xi\overline{\xi}}{\|\xi\|^2}\pi(Z)\right) \\
&= \langle x, \pi(Z)\rangle \\
&= \langle\pi^*(x), Z\rangle
\end{aligned}$$

where $\pi^* : \mathfrak{u}(V)^* \to \mathfrak{g}^*$ is dual to the representation $\pi : \mathfrak{g} \to \mathfrak{u}(V)$. In other words,

$$\Phi(\xi) = \pi^*\left(\frac{\xi\overline{\xi}}{\|\xi\|^2}\right).$$

This is constant on fibres of $p : L^\times \to \mathbb{P}(V)$, hence descends to a moment map on $\mathbb{P}(V)$, also denoted by Φ and $\Phi(x) = \pi^*(x)$. This restricts to a moment map on $W = G(\lambda)$, again denoted by Φ:

$$\langle\Phi(w), Z\rangle = \frac{1}{i}\mathrm{Tr}(w\pi(Z)) = \frac{(v, Zv)}{i\|v\|^2} \qquad (v \in \mathrm{Im}(w)).$$

This moment map is equivariant. Indeed

$$\begin{aligned}\langle\Phi(g\xi), Z\rangle &= \frac{(g\xi, Zg\xi)}{i\|g\xi\|^2}\\ &= \frac{(\xi, g^{-1}Zg\xi)}{i\|\xi\|^2}\\ &= \langle\Phi(\xi), g^{-1}Zg\rangle\\ &= \langle g(\Phi(\xi)), Z\rangle.\end{aligned}$$

So $\Phi : W \to \mathfrak{g}^*$ is a covering map onto a coadjoint orbit, namely $G(\lambda)$ since we have $\Phi(w_0) = \lambda$. Indeed:

1) $\Phi(w_0) \in \mathfrak{t}^* = (\mathfrak{g}^*)^T$. To verify it, take t in T such that $t = e^H$, then

$$\begin{aligned}t(\Phi(w_0)) = \Phi(t(w_0)) &= \pi^*\left(\frac{tv_0\overline{tv_0}}{\|tv_0\|^2}\right) = \pi^*\left(\frac{e^{i\langle\lambda,H\rangle}v_0\overline{e^{i\langle\lambda,H\rangle}v_0}}{\|e^{i\langle\lambda,H\rangle}v_0\|^2}\right)\\ &= \pi^*\left(\frac{v_0\overline{v_0}}{\|v_0\|^2}\right) = \Phi(w_0)\end{aligned}$$

(T-invariant).

2) For $H \in \mathfrak{t}$

$$\langle\Phi(w_0), H\rangle = \frac{(v_0, Hv_0)}{i\|v_0\|^2} = \frac{(v_0, i\langle\lambda, H\rangle v_0)}{i\|v_0\|^2} = \langle\lambda, H\rangle,$$

hence $\Phi(w_0) = \lambda$.

Finally, this covering is trivial (a diffeomorphism) because it's a covering of a simply connected orbit (Theorem 4.8).

5.4 The space $H^0(L)$ of antiholomorphic sections

Again, we now have diffeomorphism $\Phi : W \to X$ identifying the submanifold $W = G(w_0)$ of $\mathbb{P}(V)$ with the coadjoint orbit $X = G(\lambda)$. Denote its inverse by $I : X \to W$. We have the following diagram:

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow & & \\ X & \xrightarrow{I} & W & \subset & \mathbb{P}(V)\end{array}$$

We can pull-back the bundle $L = \{(w, \xi) \in W \times V : \xi \in \mathrm{Im}(w)\}$ and its connection $\varpi(\delta\xi) = \dfrac{(\xi, \delta\xi)}{i\|\xi\|^2}$ which satisfies $d\varpi = p^*\sigma$. Let us denote this pull-back again by $p : L \to X$. Then $L^\times$ is a complex submanifold of $V \setminus \{0\}$.

We are going to identify V with the space $H^0(L)$ of antiholomorphic sections of L. A section of L can be defined

- either as a map $s : X \to L$ such that

$$p(s(w)) = w$$

- or as a map $f : L^\times \to \mathbb{C}$ such that

$$f(z\xi) = \bar{z}f(\xi).$$

with the relation

$$s(w) = \frac{\xi f(\xi)}{\|\xi\|^2}, \quad \xi \in \mathrm{Im}(w)$$

and conversely:

$$f(\xi) = (\xi, s(w))$$

which is antilinear in its first entry ξ.

This is called *antiholomorphic*, if f is antiholomorphic, i.e. $Df(\xi) : T_\xi L^\times \to \mathbb{C}$ is complex antilinear for each ξ.

We make the space $H^0(L)$ into a (pre-)Hilbert space by putting

$$(f, f') = \int_X \frac{\overline{f(\xi)}f'(\xi)}{\|\xi\|^2} dx$$

(the integrand depends only on x, $\xi \in \mathrm{Im}(x)$), where dx is the volume form on X coming from $\sigma \wedge \ldots \wedge \sigma$ (the number of factors is dim(X)/2), normalized so that $\int_X dx = \dim(V)$ — the so-called *Liouville measure on* X.

Lemma 5.3. *Point evaluations on* $H^0(L)$ *are continuous. (Point evaluation at* ξ *is the linear form* $f \mapsto f(\xi)$*)* $\forall \xi \exists K_\xi$ *s.t.*

$$|f(\xi)| \leqslant K_\xi \|f\|.$$

Proof. See Remark 6.4. □

Note, G acts on $H^0(L)$ by $(gf)(\xi) = f(\bar{g}\xi)$ (naturally and unitarily).

Theorem 5.4 (Borel-Weil, non-constructive proof). $H^0(L)$ *is isomorphic to* V.

Proof. Define a map

$$\begin{aligned} V &\to H^0(L) \\ \varphi &\mapsto f \end{aligned}$$

by

$$f(\xi) = (\xi, \varphi),$$

where $(\cdot, \cdot)$ is the scalar product on V.

1. This map is well-defined, i.e. f is antiholomorphic in ξ:

$$Df(\xi)(\delta\xi) = (\delta\xi, \varphi)$$

is indeed antilinear.

2. This map is isometric (hence injective). Schur's Lemma (2.17) gives us

$$A := \int_X \frac{\xi(\xi, \cdot)}{\|\xi\|^2} dx = \int_X x dx = \lambda \cdot 1,$$

because

$$gAg^{-1} = \int_X g(x) dx = \int_X x dx = A.$$

(dx is invariant on X under the G-action). Taking trace

$$\mathrm{Tr}(A) = \int_X \frac{\mathrm{Tr}(\xi(\xi, \cdot))}{\|\xi\|^2} dx = \lambda \dim(V).$$

We have chosen dx such that

$$\int_X dx = \dim(V),$$

so $\lambda = 1$. Now with $f(\xi) = (\xi, \varphi)$ we get

$$\begin{aligned}\|f\|^2 &= \int_X \frac{\overline{f(\xi)} f(\xi)}{\|\xi\|^2} dx = \int_X \frac{(\varphi, \xi)(\xi, \varphi)}{\|\xi\|^2} dx \\ &= (\varphi, \int_X \frac{\xi(\xi, \cdot)}{\|\xi\|^2} dx \varphi) = \|\varphi\|^2.\end{aligned}$$

Thus, $\varphi \mapsto f = (\cdot, \varphi)$ preserves norm.

3. The map $\varphi \mapsto f$ is onto $H^0(L)$.

Indeed, by the previous Lemma 5.3 and the Riesz representation theorem: For each $\xi \in L^\times$ there is a vector $e_\xi \in H^0(L)$ such that

$$f(\xi) = (e_\xi, f) \qquad \forall f \in H^0(L).^2$$

Now,

$$(e_\xi, f) = \int_X \frac{\overline{e_\xi(\eta)} f(\eta)}{\|\eta\|^2} dy = \int_X \frac{(e_\xi, e_\eta)}{\|\eta\|^2} f(\eta) dy.$$

(The function $K(\xi, \eta) = (e_\xi, e_\eta)$ is called the *reproducing kernel* of $H^0(L)$.)

[2] Inner product in $H^0(L)$.

We are going to show that

$$(e_\xi, e_\eta)_{H^0(L)} = (\xi, \eta)_V.$$

This will finish the proof since

$$f(\xi) = \int_X \frac{(\xi,\eta)}{\|\eta\|^2} f(\eta)\,dy = (\xi, \varphi),$$

where

$$\varphi = \int_X \frac{\eta f(\eta)}{\|\eta\|^2} dy.$$

Observe:
1) $e_{g\xi} = ge_\xi \qquad \forall g \in G$,
2) $e_{z\xi} = ze_\xi \qquad \forall z \in S^1$.
Proof of 1)

$$(e_{g\xi}, f) = f(g\xi) = (\bar{g}f)(\xi) = (e_\xi, \bar{g}f) = (ge_\xi, f) \qquad \forall f \in H^0(L).$$

Proof of 2)

$$(e_{z\xi}, f) = f(z\xi) = \bar{z}f(\xi) = \bar{z}(e_\xi, f) = (ze_\xi, f) \qquad \forall f \in H^0(L).$$

Claim 5.5. $(e_\xi, e_\xi) = c\|\xi\|^2$ *for some* $c > 0$.

Proof. Consider the function on the orbit $\Phi(x) = \frac{(e_\xi, e_\xi)}{\|\xi\|^2}$, for $\xi \in Im(x)$. Now $\Phi(g(x)) = \frac{(e_{g\xi}, e_{g\xi})}{\|g\xi\|^2} = \frac{(ge_\xi, ge_\xi)}{\|g\xi\|^2} = \Phi(x)$, for all $g \in G$. Hence, $\Phi(x)$ is constant. □

Consequence of the claim:

$$(e_\xi, e_\eta) = c(\xi, \eta), \quad \forall \xi, \eta.$$

This holds because the LHS $(e_\xi, e_\eta) = e_\eta(\xi) = \overline{e_\xi(\eta)}$ is antiholomorphic in ξ and holomorphic in η. At the same time the RHS (ξ, η) is antiholomorphic in ξ and holomorphic in η too. Hence these functions are equal by analytic continuation [**B67**, 5.14.7].

Now, $c \neq 0$, because if

$$f(\xi) = \int_X \frac{(e_\xi, e_\eta)}{\|\xi\|^2} f(\eta)\,dy = 0, \qquad \forall \xi$$

then $H^0(L) = 0$. Which is not true, because we have already put V injectively there. Finally, $c = 1$:

$$\begin{aligned} c\|\xi\|^2 &= (e_\xi, e_\xi) \\ &= \int_X \frac{\overline{e_\xi(\eta)} e_\xi(\eta)}{\|\eta\|^2} dy \\ &= \int_X \frac{(e_\xi, e_\eta)(e_\eta, e_\xi)}{\|\eta\|^2} dy \\ &= c^2 \int_X \frac{(\xi, \eta)(\eta, \xi)}{\|\eta\|^2} dy \\ &= c^2(\xi, \xi) \qquad \text{(by Schur's Lemma)} \\ &= c^2\|\xi\|^2. \end{aligned}$$

Together $c = c^2$ and $c \neq 0$ give us $c = 1$. This ends the proof of the Borel-Weil Theorem 5.4. □

5.5 Realization in the space $H^0(\lambda)$ of functions on $G_{\mathbb{C}}$

In what follows it will be convenient to realize our representations more uniformly in spaces of antiholomorphic functions on the group $G_{\mathbb{C}}$ itself. To this end, consider $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm = \oplus_{\alpha \in R^\pm} \mathfrak{g}^\alpha$. We define the *Borel subalgebra* $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$ and the resulting *standard Borel subgroup* $B = \exp(\mathfrak{b})$. As we can see from Claims 1-2 in Theorem 5.2, B acts trivially on w_0. Then the action

$$bv_0 = \chi(b)v_0, \quad \chi(b) \in \mathbb{C}^\times \tag{5.1}$$

gives us a character on B.[3] The derived action defines an infinitesimal character of $\mathfrak{b}$:

$$Zv_0 = i\langle\lambda, Z\rangle v_0 = i\langle\lambda, H\rangle v_0 \tag{5.2}$$

where $Z = H + U \in \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \subset \mathfrak{g}_{\mathbb{C}}$ and we recall that $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$ extends to $\mathfrak{g}_{\mathbb{C}}$ by complex linearity. In addition to our space

$$H^0(L) = \left\{ f : L^\times \to \mathbb{C} \quad : \quad \begin{array}{l} f \text{ is antiholomorphic} \\ f(z\xi) = \bar{z} f(\xi) \quad \forall z \in \mathbb{C}^\times \end{array} \right\},$$

we now consider the space

$$H^0(\lambda) = \left\{ F : G_{\mathbb{C}} \to \mathbb{C} \quad : \quad \begin{array}{l} F \text{ is antiholomorphic} \\ F(gb) = \overline{\chi(b)} F(g) \quad \forall b \in B \end{array} \right\}.$$

[3] It's actually a character; indeed $(bb')v_0 = \chi(b)\chi(b')v_0$ and $\chi(bb') = \chi(b)\chi(b')$.

Claim 5.6. $H^0(L)$ *is isomorphic to* $H^0(\lambda)$ *with* G-*action*[4]

$$(gF)(g') = F(\bar{g}g')$$

by

$$f \mapsto F, \text{ where } F(g) = f(gv_0).^5$$

Proof. Let us denote the above map by $\Phi : H^0(L) \to H^0(\lambda)$, then

$$\begin{aligned} &\Phi : f \mapsto F \\ &F(g) = f(gv_0), \text{ for } g \in G_{\mathbb{C}}. \end{aligned}$$

First of all, Φ indeed maps $H^0(L)$ to $H^0(\lambda)$. To check this consider the derivative

$$\begin{aligned} DF(g)(i\delta g) = [\delta g = Zg] &= \frac{d}{dt}F(e^{itZ}g)|_{t=0} \\ &= \frac{d}{dt}f(e^{itZ}gv_0)|_{t=0} = Df(gv_0)(iZgv_0) \\ &= -iDf(gv_0)(Zgv_0) \quad (f \text{ is antiholomorphic}) \\ &= -iDF(g)(Zg) = -iDF(g)(\delta g). \end{aligned}$$

Hence, F is antiholomorphic. Also, (5.1) gives us

$$F(gb) = f(gbv_0) = f(g\chi(b)v_0) = \overline{\chi(b)}f(gv_0) = \overline{\chi(b)}F(g).$$

Thus F is in $H^0(\lambda)$.

Second, Φ is an intertwining map between $H^0(L)$ and $H^0(\lambda)$. Indeed:

(a) Consider $\Phi(gf) \in H^0(\lambda)$. We have $\Phi(gf)(g') = (gf)(g'v_0) = f(\bar{g}g'v_0)$.

(b) At the same time, $g(\Phi f)(g') = (gF)(g') = F(\bar{g}g') = f(\bar{g}g'v_0)$.

Together with the fact that $H^0(\lambda)$ is irreducible as will be proved presently (Theorem 6.2) this completes the proof. ☐

[4] We write $g \mapsto \overline{g}$ for the antiautomorphism of $G_{\mathbb{C}}$ whose differential at the identity is -1 on $\mathfrak{g}$ and 1 on $i\mathfrak{g}$. Note that for $g \in G$ we have $\overline{g} = g^{-1}$; our formula then extends the action to $G_{\mathbb{C}}$.

[5] Here by gv_0 we mean the $G_{\mathbb{C}}$-action on the line bundle $L^\times \subset V^\times$.

CHAPTER 6
CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM

Our goal now is to prove the following theorem:

Theorem 6.1 (Borel-Weil construction). *Assume* G *is one of the classical groups* $U(n)$, $SO(n)$, *or* $Sp(2n)$. *Let* λ *be a dominant integral element of* $\mathfrak{t}^*$. *Then* $H^0(\lambda)$ *is nonzero and forms an irreducible* G*-module with highest weight* λ.

We have

$$H^0(\lambda) = \left\{ F : G_{\mathbb{C}} \to \mathbb{C} \quad : \quad \begin{array}{l} F \text{ is antiholomorphic} \\ F(gb) = \overline{\chi(b)}F(g) \quad \forall b \in B \end{array} \right\} \tag{6.1}$$

together with

$$\chi : B \to \mathbb{C}.$$

Our steps to prove Borel-Weil:

- $H^0(\lambda)$ is irreducible or zero
- $H^0(\lambda)$ is non-zero (in a constructive way).

6.1 $H^0(\lambda)$ is irreducible or zero

We start by making $H^0(\lambda)$ into a *unitary* G-module, as follows. The functions in (6.1) are sections of the bundle $G_{\mathbb{C}} \times_B \mathbb{C} \to G_{\mathbb{C}}/B$ associated to the character χ of B. The Iwasawa decomposition $G_{\mathbb{C}} = GB$ [**B04**, p. 203] allows us to identify its (complex) base $G_{\mathbb{C}}/B$ with the (compact) manifold $G/(G \cap B) = G/T$, and thus our bundle with the bundle

$$G \times_T \mathbb{C} \to G/T$$

associated to the character χ restricted to T. Sections of it (thought of as functions $F : G \to \mathbb{C}$ such that $F(gt) = \overline{\chi(t)}F(g)$ then carry a natural inner product given by

$$(F, F') = \int_{G/T} \overline{F}F' \mathrm{vol} \tag{6.2}$$

where vol is the (unique up to scale) G-invariant volume form on G/T. This inner product is invariant under the action (5.6)

$$(gF)(g') = F(\overline{g}g'), \text{ for } F \in H^0(\lambda) \text{ and } g \in G.$$

Using the Kobayashi-Kunze argument [**K61**; **K62**] we prove the next

Theorem 6.2. $H^0(\lambda)$ *is irreducible or zero.*

Proof. First we need the following lemma:

Lemma 6.3. *Point evaluations on* $H^0(\lambda)$ *are continuous.*

Proof. The *point evaluations* are the functionals $A_g : H^0(\lambda) \to \mathbb{C}$ defined by

$$A_g(F) = F(g)$$

where $F \in H^0(\lambda)$ and $g \in G_{\mathbb{C}}$.

It is sufficient to prove that point evaluations functionals are bounded. Take g in $G_{\mathbb{C}}$. We want to prove that

$$|F(g)| \leqslant C_g \|F\|, \quad \forall F \in H^0(\lambda).$$

Let (U, φ) be a holomorphic chart of $G/T = G_{\mathbb{C}}/B$ such that $gT \in U$. Then φ maps $U \to \varphi(U) \subset \mathbb{C}^n$ and we have

$$\|F\|^2 = \int_{G/T} |F(uT)|^2 \mathrm{vol} \tag{6.3}$$

$$\geqslant \int_{U} |F(uT)|^2 \mathrm{vol} \tag{6.4}$$

$$= \int_{\varphi(U)} (\varphi^{-1})^*(|F(uT)|^2 \mathrm{vol}) \tag{6.5}$$

Now, $(\varphi^{-1})^*(|F(uT)|^2\mathrm{vol})$ is the pull-back of the top-degree form $|F(uT)|^2\mathrm{vol}$, hence

$$\begin{aligned}(\varphi^{-1})^*(|F(uT)|^2\mathrm{vol}) &= |F \circ \varphi^{-1}(z)|^2[(\varphi^{-1})^*\mathrm{vol}] \\ &= |F \circ \varphi^{-1}(z)|^2 p(z) dz,\end{aligned}$$

where $p(z)$ is a function on $\varphi(U)$, either positive or negative everywere. Now, (6.5) becomes

$$\|F\|^2 \geqslant \int_U |F(uT)|^2 \mathrm{vol} = \int_{\varphi(U)} |F \circ \varphi^{-1}(z)|^2 p(z) dz$$

and the last equality forces $p(z)$ to be positive on $\varphi(U)$.

Suppose $z \in \varphi(U)$ is the image of gT under φ : $\varphi(gT) = z$. Let D be a ball around z contained in $\varphi(U)$. Denote $F \circ \varphi^{-1}$ by $\tilde{F}$. The application of the Mean Value Theorem for the harmonic function $\tilde{F}$ [**A01**, p. 6; **F00**, p. 8] together with the Cauchy-Schwarz inequality gives us

$$\begin{aligned}|\tilde{F}(z)| &\leqslant \frac{1}{\mathrm{Vol}(D)} \int_D |\tilde{F}(t)| dt = \frac{1}{\mathrm{Vol}(D)} \int_D \frac{1}{\sqrt{p(t)}} |\tilde{F}(t)| \sqrt{p(t)} dt \\ &\leqslant \frac{1}{\mathrm{Vol}(D)} \left(\int_D \frac{1}{p(t)} dt \right)^{1/2} \left(\int_D |\tilde{F}(t)|^2 p(t) dt \right)^{1/2} \leqslant C_z \|F\|\end{aligned}$$

which completes the proof of the lemma. ☐

Remark 6.4. Arguing in the same manner, we obtain the continuity of point evaluations in the space $H^0(L)$ (Lemma 5.3).

Now we shall prove Theorem 6.2. If $H^0(\lambda) = 0$ we are done. Else, we argue much as in the proof of Theorem 5.4, part 3: Consider the G-invariant inner product $(\cdot,\cdot)$ on $H^0(\lambda)$, given by (6.2). By the Riesz representation theorem, for every $z \in G_{\mathbb{C}}$ there exists a unique vector $E_z \in H^0(\lambda)$ such that

$$F(z) = (E_z, F), \quad \forall F \in H^0(\lambda). \tag{6.6}$$

For $w \in G_{\mathbb{C}}$ that gives us

$$E_z(w) = (E_w, E_z). \tag{6.7}$$

For all $z, w \in G_{\mathbb{C}}$

$$E_z(w) = \bar{E}_w(z). \tag{6.8}$$

Now, for $g \in G_{\mathbb{C}}$ one has

$$E_{gz} = gE_z. \tag{6.9}$$

Indeed, for all $F \in H^0(\lambda)$,

$$(E_{gz}, F) = F(gz) = (\bar{g}F)(z) = (E_z, \bar{g}F) = (gE_z, F).$$

Hence $E_{gz} = gE_z$. Suppose that H' is a nonzero closed subspace of $H^0(\lambda)$ invariant under the G-action. Let E'_z be the orthogonal projection of E_z on H'. Then E'_z satisfies analogs of (6.6–6.9). Consider the function $\Phi(z) = \frac{E'_z(z)}{E_z(z)}$. Using the Iwasawa decomposition [**B04**, p. 203] $G_{\mathbb{C}} = GB$ together with the fact that G acts unitarily (but not $G_{\mathbb{C}}$), we get:

For $g \in G_{\mathbb{C}}, g = ub$, where $u \in G$, $b \in B$,

$$\begin{aligned}\Phi(g) &= \frac{E'_g(ub)}{E_g(ub)} = \frac{\overline{\chi(b)}E'_g(u)}{\overline{\chi(b)}E_g(u)} = \frac{\overline{E'_u(g)}}{\overline{E_u(g)}} = \frac{\overline{E'_u(ub)}}{\overline{E_u(ub)}} \\ &= \frac{\overline{\chi(b)}E'_u(u)}{\overline{\chi(b)}E_u(u)} = \frac{E'_u(u)}{E_u(u)} = \frac{(uE'_e, uE'_e)}{(uE_e, uE_e)} = \frac{(E'_e, E'_e)}{(E_e, E_e)} = \Phi(e)\end{aligned}$$

so $\Phi(z)$ is constant. That gives us

$$E'_z(z) = cE_z(z), \quad \forall z \in G_{\mathbb{C}}. \tag{6.10}$$

For fixed z both $E'_z(w)$ and $cE_z(w)$ are antiholomorphic as functions of w and by (6.8) both are holomorphic functions of z, w being fixed. Since they coincide on the diagonal $z = w$, it follows by analytic continuation that they coincide everywhere [**B67**, 5.14.7]. Therefore (6.10) becomes

$$E'_z(w) = cE_z(w).$$

The condition above easily gives us $H' = H^0(\lambda)$ since any F orthogonal to H' in $H^0(\lambda)$ is orthogonal to E'_z for every z and hence is zero by (6.6). Thus $H^0(\lambda)$ is irreducible. □

6.2 $H^0(\lambda)$ is non-zero

In this section we consider a compact Lie group G together with a dominant integral form $\lambda \in \mathfrak{t}^*$. We want to show that the resulting space $H^0(\lambda)$ does not reduce to the zero function.

6.2.1 Heuristics and $U(n)$

Heuristics 1: Instead of looking for an arbitrary nonzero $F \in H^0(\lambda)$ we will focus on finding a (nonzero) *highest weight vector* $F \in H^0(\lambda)$. Then F is going to satisfy

$$uF = F \quad \forall u \in N^+ = exp(\oplus_{\alpha>0}\mathfrak{g}^\alpha).$$

Now,

$$\begin{aligned} uF &= F \\ (uF)(g) &= F(g) \\ F(\overline{u}g) &= F(g). \end{aligned}$$

Hence we will have

$$F(\overline{u}b) = F(b) = \overline{\chi(b)}F(1)$$

for all $u \in N^+$ and $b \in B$. Picking $F(1) = 1$ defines it entirely in N^-B:

$$F(\overline{u}b) = \overline{\chi(b)}. \tag{6.11}$$

But N^-B is known to be open dense in $G_\mathbb{C}$ (by the Bruhat decomposition recalled in the Appendix), so our task is to show that (6.11) extends from N^-B to an antiholomorphic function F on all $G_\mathbb{C}$.

Heuristics 2: Starting with $G = U(n)$ we consider its complexification $G_\mathbb{C} = GL(n, \mathbb{C})$. The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of $G_\mathbb{C} = GL(n, \mathbb{C})$ has a root space decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+$$

Here $\mathfrak{t}_\mathbb{C}$ consists of all diagonal matrices. The subalgebras $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are of the following form:

$$\mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\}, \mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

We denote $\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$ by $\mathfrak{b}$, that gives us

$$\mathfrak{b} = \left\{ \begin{pmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \right\}.$$

Recall that the maximal torus in $U(n)$ consists of all diagonal matrices in $U(n)$. From $\mathfrak{n}^-$ and $\mathfrak{b}$ we get N^-

$$N^- = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

and the Borel subgroup B:

$$B = \left\{ \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} \right\}.$$

Under the identification (4.2), the dominant integral elements $\lambda \in \mathfrak{t}^*$ take the form

$$\lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with the conditions $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. That gives us a character χ on T in the following way. Write H for the element in $\mathfrak{t}$,

$$H = \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix}$$

so that $\langle \lambda, H \rangle = \lambda_1\theta_1 + \cdots + \lambda_n\theta_n$. Then $\chi(exp(H)) = e^{i\langle \lambda, H \rangle} = (e^{i\theta_1})^{\lambda_1} \dots (e^{i\theta_n})^{\lambda_n}$ or

$$\chi : \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \mapsto (z_1)^{\lambda_1} \dots (z_n)^{\lambda_n}. \tag{6.12}$$

As soon as we are going to extend the character χ from T to B and then to a function $\bar{F}$ on N^-B and eventually on all of $G_{\mathbb{C}}$, we need to use a form other than (6.12) for

χ. Putting $l_i = \lambda_i - \lambda_{i+1}$ (with $\lambda_{n+1} := 0$) we have

$$\lambda = \begin{pmatrix} l_1 + \cdots + l_n & & & \\ & l_2 + \cdots + l_n & & \\ & & \ddots & \\ & & & l_n \end{pmatrix}$$

with $l_1, l_2, \ldots, l_{n-1} \in \mathbb{Z}_+$ and $l_n \in \mathbb{Z}$, and (6.12) becomes

$$\chi : \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \mapsto (z_1)^{l_1}(z_1z_2)^{l_2}\ldots(z_1z_2\ldots z_n)^{l_n},$$

where $l_1, l_2, \ldots, l_{n-1}$ are non-negative integers and l_n a (possibly negative) integer. As in (5.2) this extends to a character on B. This defines the function (6.11) (highest weight vector) on the big cell $N^-B \subset GL(n, \mathbb{C})$ as follows:

$$F(ldu) = F(lb) = \overline{\chi(b)} = \overline{\chi(d)} \tag{6.13}$$

for $l \in N^-$, $d \in T_{\mathbb{C}}$, and $u \in N^+$. Explicitly, in $GL(n, \mathbb{C})$, this map is

$$F : ldu = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ & & & \ddots & \\ l_{n1} & l_{n2} & & & 1 \end{pmatrix} \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & & u_{1n} \\ & 1 & u_{23} & & u_{2n} \\ & & 1 & & \\ & & & \ddots & u_{n-1n} \\ & & & & 1 \end{pmatrix}$$

$$\mapsto \overline{(d_1)^{l_1}(d_1d_2)^{l_2}\ldots(d_1d_2\ldots d_n)^{l_n}}.$$

We have defined the function F on the "big cell" $N^-B \subset G_{\mathbb{C}}$ and our task is to show that it extends to an antiholomorphic function on all of $G_{\mathbb{C}}$. We know by the Bruhat decomposition (reviewed in the Appendix) that Weyl group translates $\dot{w}N^-B$ of the big cell make a covering of $G_{\mathbb{C}}$ by open dense sets. So it is enough to show that (6.13) extends to each translate.

Heuristics 3: Consider the case $G_{\mathbb{C}} = GL(4, \mathbb{C})$. As we know on the big cell N^-B the function F sends

$$ldu = \begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & c & 1 & \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}$$

to

$$\overline{(x)^{l_1}(xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}}.$$

We consider different translates $\dot{w}ldu$ of the element ldu, assuming they belong to the big cell, which is open dense in $G_{\mathbb{C}}$. Hereafter R.E.F. stands for row echelon form: $\mathrm{R.E.F.}(ldu) = du$. Note that under our assumptions, $\dot{w}l$ is in the big cell too, so we can decompose it as $\dot{w}l = l_1d_1u_1$. Now $F(\dot{w}ldu) = F(l_1d_1u_1du) = F(d_1u_1du) = F(\mathrm{R.E.F.}(\dot{w}l)du)$ for $\dot{w}ldu$ from the big cell N^-B.

• Case $w = (12)$: then

$$\begin{aligned}
F(\dot{w}ldu) &= F\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & c & 1 & \\ d & e & f & 1 \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right) \\
&= F\left(\mathrm{R.E.F.}\left(\begin{pmatrix} a & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix}\right)\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right) \\
&= F\left(\begin{pmatrix} a & 1 & 0 & 0 \\ & a^{-1} & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right) \\
&= F\begin{pmatrix} ax & * & * & * \\ & a^{-1}y & * & * \\ & & z & * \\ & & & t \end{pmatrix} \\
&= \overline{(ax)^{l_1}(xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}}
\end{aligned}$$

• Case $w = (123) = (12)(23)$:

$$\begin{aligned}
F(\dot{w}ldu) &= F\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & c & 1 & \\ d & e & f & 1 \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right) \\
&= F\left(\mathrm{R.E.F.}\left(\begin{pmatrix} a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ 1 & 0 & 0 & 0 \\ d & e & f & 1 \end{pmatrix}\right)\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right)
\end{aligned}$$

$$= F\left(\begin{pmatrix} a & 1 & 0 & 0 \\ & c-a^{-1}b & 1 & 0 \\ & & (c-a^{-1}b)^{-1}a^{-1} & 0 \\ & & & 1 \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right)$$
$$= F\begin{pmatrix} ax & * & * & * \\ & (c-a^{-1}b)y & * & * \\ & & (c-a^{-1}b)^{-1}a^{-1}z & * \\ & & & t \end{pmatrix}$$
$$= \overline{(ax)^{l_1}((ac-b)xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}}$$

- Case $w = (1234) = (12)(23)(34)$:

$$F(\dot{w}ldu) = F\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & c & 1 & \\ d & e & f & 1 \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right)$$
$$= F\left(\mathrm{R.E.F.}\left(\begin{pmatrix} a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right)$$
$$= F\left(\begin{pmatrix} a & 1 & 0 & 0 \\ & c-a^{-1}b & 1 & 0 \\ & & f-(c-a^{-1}b)^{-1}(e-a^{-1}d) & 1 \\ & & & * \end{pmatrix}\begin{pmatrix} x & & & \\ & y & & \\ & & z & \\ & & & t \end{pmatrix}\begin{pmatrix} 1 & a' & b' & c' \\ & 1 & d' & e' \\ & & 1 & f' \\ & & & 1 \end{pmatrix}\right)$$
$$= F\begin{pmatrix} ax & * & * & * \\ & (c-a^{-1}b)y & * & * \\ & & (f-(c-a^{-1}b)^{-1}(e-a^{-1}d))z & * \\ & & & * \end{pmatrix}$$
$$= \overline{(ax)^{l_1}((ac-b)xy)^{l_2}((acf-ae-bf+d)xyz)^{l_3}(xyzt)^{l_4}}$$

$$= \overline{a^{l_1}\begin{vmatrix} a & 1 \\ b & c \end{vmatrix}^{l_2}\begin{vmatrix} a & 1 & 0 \\ b & c & 1 \\ d & e & f \end{vmatrix}^{l_3}(x)^{l_1}(xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}}.$$

We can see that in each case we have

$$F(\dot{w}g) = F(\dot{w}ldu) = \overline{\det_1(\dot{w}l)^{l_1}\det_2(\dot{w}l)^{l_2}\det_3(\dot{w}l)^{l_3}x^{l_1}(xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}},$$

where $\det_i(m)$ denotes the i-th principal minor of a matrix m. Observe that $\det_i(\dot{w}ldu) = \det_i(\dot{w}l)\det_i(d)$. Indeed, writing matrices in block form, where the first block has size $i \times i$, we have

$$\begin{aligned}\dot{w}ldu &= \begin{pmatrix} (\dot{w}l)_1 & (\dot{w}l)_2 \\ (\dot{w}l)_3 & (\dot{w}l)_4 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} u_1 & * \\ 0 & * \end{pmatrix} \\ &= \begin{pmatrix} (\dot{w}l)_1 & (\dot{w}l)_2 \\ (\dot{w}l)_3 & (\dot{w}l)_4 \end{pmatrix} \begin{pmatrix} d_1u_1 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} (\dot{w}l)_1 d_1 u_1 & * \\ * & * \end{pmatrix}\end{aligned}$$

This confirms $\det_i(\dot{w}ldu) = \det_i(\dot{w}l)\det_i(d)$.

After this consideration, $F(\dot{w}g)$ above becomes $F(\dot{w}g) = \overline{\det_1(\dot{w}g)^{l_1} \ldots \det_n(\dot{w}g)^{l_n}}$ or in other words, renaming $\dot{w}g$ as g,

$$F(g) = \overline{\det_1(g)^{l_1}\det_2(g)^{l_2} \ldots \det_n(g)^{l_n}}. \tag{6.14}$$

Now, taking $F : G_{\mathbb{C}} \to \mathbb{C}$ of the form (6.14) we get an antiholomorphic function that clearly extends F from the big cell. This concludes our heuristics and we state:

Theorem 6.5. *$H^0(\lambda)$ is non-zero for the $U(n)$ case.*

Proof. We check that the function (6.14) is really in $H^0(\lambda)$. Indeed, by the same block multiplication argument as above we have

$$\det_i(gb) = \det_i(gd'u') = \det_i(gd') = \det_i(g)\det_i(d') = \det_i(g)\det_i(b).$$

Hence

$$\begin{aligned}F(gb) = \overline{\prod_{i=1}^{n} \det_i(gb)^{l_i}} &= \overline{\prod_{i=1}^{n} (\det_i(g)\det_i(b))^{l_i}} \\ &= \overline{\chi(b)\prod_{i=1}^{n} \det_i(g)^{l_i}} = \overline{\chi(b)}F(g).\end{aligned}$$

We have constructed a nonzero vector F in the G-module $H^0(\lambda)$. □

Remark 6.6. The function F in (6.14) is indeed antiholomorphic, as the conjugate of a polynomial map. Moreover, if $\lambda \in \mathfrak{t}^*$ is not dominant, then some of the l_i in (6.14) would be negative, forcing F (or even the character (6.12)) to not be (anti)holomorphic. Hence, if λ is not dominant, then the space $H^0(\lambda)$ is zero. Also note that Theorems 6.2 and 2.18 imply that $H^0(\lambda)$ is finite-dimensional.

6.2.2 $Sp(2n)$

Let J denote the $n \times n$ matrix $\begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{pmatrix}$. Let $M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$. Define $Sp(2n, \mathbb{C}) = \{g \in SL(2n, \mathbb{C}) \, : \, {}^t g M g = M\}$. In other words, $Sp(2n, \mathbb{C})$ is the set of fixed points of the involution σ of $SL(2n, \mathbb{C})$:

$$Sp(2n, \mathbb{C}) = \{g \in SL(2n, \mathbb{C}) : \sigma(g) = g\}, \tag{6.15}$$

given by $\sigma(g) = M({}^t g^{-1})M^{-1}$.

In this setting, the maximal torus T of $Sp(2n) = Sp(2n, \mathbb{C}) \cap U(2n)$ consists of the following matrices

$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_n} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_1} \end{pmatrix}, \theta_i \in \mathbb{R}$$

together with the Cartan subalgebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & & & & \\ & \ddots & & & & \\ & & i\theta_n & & & \\ & & & -i\theta_n & & \\ & & & & \ddots & \\ & & & & & -i\theta_1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}.$$

The groups N^- and B in $SL(2n, \mathbb{C})$ are the unipotent lower triangular and the upper triangular matrices of determinant 1. Taking their σ-fixed elements we obtain N^- and B for $Sp(2n, \mathbb{C})$:

$$N^- = \left\{ \bar{u} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} : \bar{u} \in Sp(2n, \mathbb{C}) \right\},$$

$$B = \left\{ b = \begin{pmatrix} b_1 & & & & & \\ & \ddots & & & * & \\ & & b_n & & & \\ & & & b_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & b_1^{-1} \end{pmatrix} : b \in Sp(2n, \mathbb{C}) \right\}.$$

Now, the dominant integral elements $\lambda \in \mathfrak{t}^*$ under the identification

$$\begin{aligned} i\mathfrak{g} &\to \mathfrak{g}^* \\ x &\mapsto \frac{1}{2i}\mathrm{Tr}(x\,\cdot), \end{aligned} \tag{6.16}$$

are as follows

$$\lambda = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_n & & \\ & & & & \ddots & \\ & & & & & -\lambda_1 \end{pmatrix}$$

where $\lambda_1 = l_1 + \cdots + l_n, \lambda_2 = l_2 + \cdots + l_n, \ldots, \lambda_n = l_n$ for non-negative integers l_i. That gives us a character on T as follows. For $H \in \mathfrak{t}$,

$$H = \begin{pmatrix} i\theta_1 & & & & & \\ & \ddots & & & & \\ & & i\theta_n & & & \\ & & & -i\theta_n & & \\ & & & & \ddots & \\ & & & & & -i\theta_1 \end{pmatrix}$$

we have $\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i$. Now, $\chi(\exp(H)) = e^{i\langle \lambda, H \rangle} = (e^{i\theta_1})^{\lambda_1} \ldots (e^{i\theta_n})^{\lambda_n}$ and on T:

$$\chi : \begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_n} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_1} \end{pmatrix} \mapsto (e^{i\theta_1})^{l_1}(e^{i\theta_1}e^{i\theta_2})^{l_2} \ldots (e^{i\theta_1}e^{i\theta_2} \ldots e^{i\theta_n})^{l_n}$$

As in (5.2) this extends to a character of B

$$\chi : \begin{pmatrix} b_1 & & & & & \\ & \ddots & & & * & \\ & & b_n & & & \\ & & & b_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & b_1^{-1} \end{pmatrix} \mapsto (b_1)^{l_1}(b_1 b_2)^{l_2}\dots(b_1 b_2 \dots b_n)^{l_n}.$$

This defines the function (6.11) (highest weight vector) on the big cell $N^- B \subset Sp(2n, \mathbb{C})$ as follows:

$$F(ldu) = F(lb) = \overline{\chi(b)} = \overline{\chi(d)} \tag{6.17}$$

for $l \in N^-$, $d \in T_{\mathbb{C}}$, and $u \in N^+$.

Arguing in the same way as for the $GL(n, \mathbb{C})$ case we obtain the antiholomorphic function $F \in H^0(\lambda)$ of the following form

$$F(g) = \overline{\det_1(g)^{l_1} \det_2(g)^{l_2} \dots \det_n(g)^{l_n}}, \tag{6.18}$$

and we prove

Theorem 6.7. $H^0(\lambda)$ *is non-zero for the* $Sp(2n)$ *case.*

6.2.3 $SO(2n+1)$

Let N be the $(2n+1)\times(2n+1)$ matrix $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdots & & \\ 1 & & & \end{pmatrix}$. Replacing M by N in the definition (6.15) of $Sp(2n, \mathbb{C})$ we obtain the definition of

$$SO(2n+1, \mathbb{C}) = \{g \in SL(2n+1, \mathbb{C}) : \sigma(g) = g\}, \tag{6.19}$$

where $\sigma(g) = N({}^t g^{-1})N^{-1}$.

In the case of $SO(2n+1) = SO(2n+1, \mathbb{C}) \cap U(2n+1)$, the maximal torus T consists of the following matrices

$$\begin{pmatrix} e^{i\theta_1} & & & & & & \\ & \ddots & & & & & \\ & & e^{i\theta_n} & & & & \\ & & & 1 & & & \\ & & & & e^{-i\theta_n} & & \\ & & & & & \ddots & \\ & & & & & & e^{-i\theta_1} \end{pmatrix}.$$

The corresponding Cartan subalgebra $\mathfrak{t}$ is

$$\left\{ \begin{pmatrix} i\theta_1 & & & & & & \\ & \ddots & & & & & \\ & & i\theta_n & & & & \\ & & & 0 & & & \\ & & & & -i\theta_n & & \\ & & & & & \ddots & \\ & & & & & & -i\theta_1 \end{pmatrix} \right\}, \tag{6.20}$$

where $\theta_i \in \mathbb{R}$. The dominant integral elements $\lambda \in \mathfrak{t}^*$ under the identification (6.16) take the form

$$\begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_n & & & & \\ & & & 0 & & & \\ & & & & -\lambda_n & & \\ & & & & & \ddots & \\ & & & & & & -\lambda_1 \end{pmatrix},$$

where $\lambda_1 = l_1 + \cdots + l_n, \lambda_2 = l_2 + \cdots + l_n, \ldots, \lambda_n = l_n$ for non-negative integers $l_1, \ldots, l_n$.

Write H for element in $\mathfrak{t}$ of the form (6.20) so that

$$\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i = (l_1 + \cdots + l_n)\theta_1 + (l_2 + \cdots + l_n)\theta_2 + \cdots + l_n \theta_n.$$

Then $\chi(exp(H)) = e^{i\langle \lambda, H\rangle}$ or

$$\chi : \begin{pmatrix} z_1 & & & & & & \\ & \ddots & & & & & \\ & & z_n & & & & \\ & & & 1 & & & \\ & & & & z_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & z_1^{-1} \end{pmatrix} \mapsto (z_1)^{l_1}(z_1 z_2)^{l_2} \ldots (z_1 z_2 \ldots z_{n-1})^{l_{n-1}}(z_1 z_2 \ldots z_n)^{l_n}.$$

This defines the function (6.11) (highest weight vector) on the big cell $N^- B \subset SO(2n+1, \mathbb{C})$ as follows:

$$F(ldu) = F(lb) = \overline{\chi(b)} = \overline{\chi(d)} \tag{6.21}$$

for $l \in N^-$, $d \in T_{\mathbb{C}}$, and $u \in N^+$.

We argue in the same way as for the $GL(n,\mathbb{C})$ case to obtain the antiholomorphic function $F \in H^0(\lambda)$ in the following form

$$F(g) = \overline{\det_1(g)^{l_1}\det_2(g)^{l_2}\ldots\det_n(g)^{l_n}}, \tag{6.22}$$

and we prove

Theorem 6.8. $H^0(\lambda)$ *is non-zero for the* $SO(2n+1)$ *case.*

6.2.4 $SO(2n)$

Let N be the $2n\times 2n$ matrix $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$. Replacing M by N in the definition (6.15) of $Sp(2n,\mathbb{C})$ we obtain the definition of

$$SO(2n,\mathbb{C}) = \{g \in SL(2n,\mathbb{C}) : \sigma(g) = g\}, \tag{6.23}$$

where $\sigma(g) = N({}^t g^{-1})N^{-1}$.

The maximal torus T of $SO(2n) = SO(2n,\mathbb{C}) \cap U(2n)$ consists of the following matrices:

$$\begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_n} & & \\ & & & & \ddots & \\ & & & & & e^{-i\theta_1} \end{pmatrix}.$$

The corresponding Cartan subalgebra $\mathfrak{t}$ is

$$\begin{pmatrix} i\theta_1 & & & & & \\ & \ddots & & & & \\ & & i\theta_n & & & \\ & & & -i\theta_n & & \\ & & & & \ddots & \\ & & & & & -i\theta_1 \end{pmatrix}, \tag{6.24}$$

where $\theta_i \in \mathbb{R}$. The dominant integral elements $\lambda \in \mathfrak{t}^*$ under the identification (6.16) take the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_n & & \\ & & & & \ddots & \\ & & & & & -\lambda_1 \end{pmatrix},$$

where $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant |\lambda_n|$. Defining $l_i = \lambda_i - \lambda_{i+1}$ for $i < n$ and $l_n = \lambda_{n-1} + \lambda_n$ we have

$$\begin{cases} \lambda_1 = l_1 + \cdots + l_{n-1} + \frac{l_n - l_{n-1}}{2} \\ \lambda_2 = l_2 + \cdots + l_{n-1} + \frac{l_n - l_{n-1}}{2} \\ \ldots \\ \lambda_{n-1} = l_{n-1} + \frac{l_n - l_{n-1}}{2} \\ \lambda_n = \frac{l_n - l_{n-1}}{2} \end{cases}$$

where $l_1, \ldots l_n$ are non-negative integers such that $l_n - l_{n-1} \in 2\mathbb{Z}$. Writing H for the element in $\mathfrak{t}$ of the form (6.24), we have $\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i$ and the character is given on T by $\chi(exp(H)) = e^{i\langle \lambda, H \rangle}$ for $H \in \mathfrak{t}$. Take the element of T of the form

$$t = \begin{pmatrix} z_1 & & & & & \\ & \ddots & & & & \\ & & z_n & & & \\ & & & z_n^{-1} & & \\ & & & & \ddots & \\ & & & & & z_1^{-1} \end{pmatrix}.$$

Now we want to construct an antiholomorphic function on $G_{\mathbb{C}}$ out of the character χ. To succeed in that we write χ in two different ways depending on the sign of $\lambda_n = \frac{l_n - l_{n-1}}{2}$. Namely,

$$\chi(t) = \begin{cases} (z_1)^{l_1}(z_1 z_2)^{l_2} \ldots (z_1 z_2 \ldots z_{n-1})^{l_{n-1}} (z_1 z_2 \ldots z_n)^{\frac{l_n - l_{n-1}}{2}} & \text{if } \lambda_n = \frac{l_n - l_{n-1}}{2} \geqslant 0, \\ \\ (z_1)^{l_1}(z_1 z_2)^{l_2} \ldots (z_1 z_2 \ldots z_{n-1})^{l_n} (z_1 z_2 \ldots z_n^{-1})^{\frac{l_{n-1} - l_n}{2}} & \text{if } \lambda_n = \frac{l_n - l_{n-1}}{2} < 0. \end{cases}$$

This defines the function (6.11) (highest weight vector) on the big cell $N^- B \subset SO(2n, \mathbb{C})$ as follows:

$$F(ldu) = F(lb) = \overline{\chi(b)} = \overline{\chi(d)} \tag{6.25}$$

for $l \in N^-$, $d \in T_{\mathbb{C}}$, and $u \in N^+$.

Arguing in the same way as for the $GL(n,\mathbb{C})$ case we obtain the antiholomorphic function $F \in H^0(\lambda)$ in the following form

$$F(g) = \begin{cases} \overline{\det_1(g)^{l_1}\det_2(g)^{l_2}\ldots\det_{n-1}(g)^{l_{n-1}}\det_n(g)^{\frac{l_n-l_{n-1}}{2}}} & \text{if } \lambda_n = \frac{l_n-l_{n-1}}{2} \geqslant 0, \\ \overline{\det_1(g)^{l_1}\det_2(g)^{l_2}\ldots\det_{n-1}(g)^{l_n}\det_n(g_0gg_0)^{\frac{l_{n-1}-l_n}{2}}} & \text{if } \lambda_n = \frac{l_n-l_{n-1}}{2} < 0, \end{cases}$$

where g_0 interchanges e_n with e_{n+1}:

$$g_0 = \begin{array}{@{}c@{}l@{}} \left(\begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) & \begin{array}{l} \\ \\ n \\ n+1. \\ \\ \\ \end{array} \end{array}$$

Lemma 6.9. *The function* $F(g)$, *as defined above, indeed extends the function* (6.25) *from the big cell.*

Proof. When $\lambda_n \geqslant 0$, direct computation and the observation that $\det_i(ldu) = d_{11}\ldots d_{ii}$ complete the proof. When $\lambda_n < 0$, the only difference is that there remains to show that $\det_n(g_0gg_0)$ extends $F(ldu) = d_{11}d_{22}...d_{nn}^{-1}$. To this end, note that the lower (resp. upper) triangular matrices $l \in N^-$ and $u \in N^+$ in $SO(2n,\mathbb{C})$ have zeros in the $(n+1,n)$, resp. $(n,n+1)$ entry (compare [**B05**, p. 212]). Indeed, take $l \in N^-$ of the form

$$l = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad \text{so} \quad l^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}.$$

We know that $\sigma(l) = l$ and this gives us $C = -DJ^tCJA$. Now let e_1, e_n be vectors from the standard basis in $\mathbb{C}^n$ and compute

$$\begin{aligned} C_{1n} &= {}^te_1Ce_n \\ &= -{}^te_1DJ^tCJAe_n \\ &= -{}^te_1J^tCJe_n \quad (\text{because } {}^te_1D = {}^te_1 \text{ and } Ae_n = e_n \text{ by triangularity}) \\ &= -{}^te_n{}^tCe_1 \\ &= -({}^tC)_{n1} \\ &= -C_{1n}. \end{aligned}$$

Hence $l_{n+1,n} = C_{1n} = 0$. Likewise (taking the transpose) $u_{n,n+1} = 0$ for $u \in N^+$ in $SO(2n, \mathbb{C})$. This observation confirms that $g_0 l g_0$ and $g_0 u g_0$ are again unipotent lower (resp. upper) triangular matrices in $SO(2n, \mathbb{C})$. Now take $g = ldu$ in the big cell N^-B of $SO(2n, \mathbb{C})$ and compute

$$\begin{aligned}
\det_n(g_0 g g_0) &= \det_n(g_0 l g_0 g_0 d g_0 g_0 u g_0) \\
&= \det_n(g_0 l g_0)\det_n(g_0 d g_0)\det_n(g_0 u g_0) \\
&= \det_n(g_0 d g_0) \\
&= d_{11} d_{22} \ldots d_{nn}^{-1}. \quad \square
\end{aligned}$$

Thus we can state our final result, which completes the proof of Theorem 6.1:

Theorem 6.10. $H^0(\lambda)$ *is non-zero for the* $SO(2n)$ *case.*

Remark 6.11. In the case $\lambda_n < 0$ we could also write

$$\chi(t) = (z_1)^{l_1}(z_1 z_2)^{l_2} \ldots (z_1 z_2 \ldots z_{n-1})^{l_{n-1}} (z_1^{-1} z_2^{-1} \ldots z_n^{-1})^{\frac{l_{n-1} - l_n}{2}}.$$

That gives us the function

$$F(g) = \overline{\det_1(g)^{l_1} \det_2(g)^{l_2} \ldots \det_{n-1}(g)^{l_{n-1}} \det_n(g^{-1})^{\frac{l_{n-1} - l_n}{2}}}$$

on $SO(2n, \mathbb{C})$ which is antiholomorphic for $\frac{l_n - l_{n-1}}{2} < 0$ and coincides with (6.25) on the big cell.

Appendix A
BRUHAT DECOMPOSITION

In this appendix we review the Bruhat decomposition of the classical complex groups, namely, $GL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$, following the exposition in [**S74**, pp. 72–75].

Let J denote the $n \times n$ matrix $\begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{pmatrix}$. Let $M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$. Define $Sp(2n, \mathbb{C})$ as the set of fixed points of the involution σ of $SL(2n, \mathbb{C})$:

$$Sp(2n, \mathbb{C}) = \{g \in SL(2n, \mathbb{C}) : \sigma(g) = g\},$$

given by $\sigma(g) = M({}^t g^{-1})M^{-1}$.

Recall that for a compact Lie group G we use T, W, and B to denote a maximal torus of G, the resulting Weyl group, and a standard Borel subgroup of $G_{\mathbb{C}}$.

Theorem A.1 (Bruhat Lemma). *Let* $G_{\mathbb{C}} = Sp(2n, \mathbb{C})$*. Then,*

(a) $G_{\mathbb{C}}$ *can be written as* $G_{\mathbb{C}} = \bigsqcup_{w \in W} BwB$*,*

(b) $BwB = Bw'B$ *iff* $w = w'$*.*

Proof. Recall that the Weyl group is $W = N(T)/T$ and denote a representative of the element $w \in W$ in $N(T)$ by $\dot{w}$. In $GL(2n, \mathbb{C})$ the Bruhat lemma follows from the Gaussian elimination process and can be stated as follows [**H92**, pp. 107–109]:

$$GL(2n, \mathbb{C}) = \bigsqcup_{w \in W} U\dot{w}B,$$

and each element of $U\dot{w}B$ is uniquely expressible in the form $u\dot{w}b$ with $b \in B$ and $u \in U_w = U \cap \dot{w}\bar{U}\dot{w}^{-1}$. (Here U and $\bar{U}$ are respectively the unipotent upper and lower triangular groups.)

Note that σ keeps B, T and $N(T)$ (of $GL(2n, \mathbb{C})$) invariant:

(a) Let $x \in B$. Then $\sigma(x) = M({}^t x^{-1})M^{-1}$. Now $x^{-1} \in B$ again, so let $x^{-1} = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$. Hence $\sigma(x) = \begin{pmatrix} J^t ZJ & -J^t YJ \\ 0 & J^t XJ \end{pmatrix}$ is in B too.

(b) Similarly, $\sigma(T) \subset T$.

(c) Let $n \in N(T)$. Then $nTn^{-1} \subset T$ gives us $\sigma(n)T\sigma(n)^{-1} \subset T$ and thus $\sigma(n) \in N(T)$.

The automorphism σ acts on U_w as follows. If $u \in U_w = U \cap \dot{w}\bar{U}\dot{w}^{-1}$ then

$$\sigma(u) \in \sigma(U) \cap \sigma(\dot{w})\sigma(\bar{U})\sigma(\dot{w})^{-1}.$$

Now, $\sigma(\dot{w}) \in N(T), \sigma(\bar{U}) \subset \bar{U}$ and

$$\sigma(u) \in U_{w'}.$$

Let g be an element in $Sp(2n, \mathbb{C})$, then

$$g = u\dot{w}b \quad \text{uniquely in } GL(2n, \mathbb{C})$$

and

$$g = \sigma(g) = \sigma(u)\sigma(\dot{w})\sigma(b).$$

Hence

$$g = \sigma(g) = u_{w'}n_{w'}b'$$

(with $u_{w'} \in U_{w'}, n_{w'} \in N(T)$ corresponding to $w' \in W, b' \in B$).

Using uniqueness of the decomposition, we get $w = w'$, $\sigma(\dot{w}) \in T\dot{w}$, $\sigma(u) = u$ and $\sigma(b) = b(\operatorname{mod} T)$. Thus from the Bruhat decomposition in $GL(2n, \mathbb{C})$ we obtain the (unique) decomposition in $Sp(2n, \mathbb{C})$. $\square$

Remark A.2. Replacing $-J$ by J in M we obtain the Bruhat Lemma and its proof for $SO(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$.

Bibliography

[**A01**] Sheldon Axler, Paul Bourdon, and Wade Ramey, *Harmonic function theory.* Graduate Texts in Mathematics, vol. 137. Springer-Verlag, New York, 2001.

[**B79**] Raoul Bott, *The geometry and representation theory of compact Lie groups.* In: Glenys L. Luke (ed.) Representation Theory of Lie Groups (Oxford, 28 June-15 July 1977), pp. 65–90. Cambridge University Press, Cambridge, 1979.

[**B67**] Nicolas Bourbaki, *Variétés différentielles et analytiques. Fascicule de résultats. Paragraphes 1 à 7.* Hermann, Paris, 1967.

[**B05**] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 7–9.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[**B04**] Daniel Bump, *Lie groups.* Graduate Texts in Mathematics, vol. 225. Springer-Verlag, New York, 2004.

[**C13**] Élie Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane.* Bull. Soc. Math. France **41** (1913) 53–96.

[**C48**] Claude Chevalley, *Sur la classification des algèbres de Lie simples et de leurs représentations.* C. R. Acad. Sci. Paris **227** (1948) 1136–1138, 1197.

[**F00**] Jacques Faraut, Soji Kaneyuki, Adam Korányi, Qi-keng Lu, and Guy Roos, *Analysis and geometry on complex homogeneous domains.* Progress in Mathematics, vol. 185. Birkhäuser Boston, Inc., Boston, MA, 2000.

[**G98**] Roe Goodman and Nolan R. Wallach, *Representations and invariants of the classical groups.* Encyclopedia of Mathematics and its Applications, vol. 68. Cambridge University Press, Cambridge, 1998. (Reprint: *Symmetry, representations, and invariants.* Springer, Dordrecht, 2009.)

[**G84**] Hans Grauert and Reinhold Remmert, *Coherent analytic sheaves.* Grundlehren der Mathematischen Wissenschaften, vol. 265. Springer-Verlag, Berlin, 1984.

[**H03**] Brian C. Hall, *Lie groups, Lie algebras, and representations: An elementary introduction.* Graduate Texts in Mathematics, vol. 222. Springer-Verlag, New York, 2003.

[**H51**] Harish-Chandra, *On some applications of the universal enveloping algebra of a semisimple Lie algebra.* Trans. Amer. Math. Soc. **70** (1951) 28–96.

[**H63**] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*, vol. 1. Springer-Verlag, Berlin, 1963.

[**H92**] Roger Howe, *A century of Lie theory.* In: Felix E. Browder (ed.) American Mathematical Society centennial publications, Vol. II (Providence, RI, 1988), pp. 101–320. Amer. Math. Soc., Providence, RI, 1992.

[**H75**] James E. Humphreys, *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[**H56**] G. A. Hunt, *A theorem of Élie Cartan*. Proc. Amer. Math. Soc. **7** (1956) 307–308.

[**J87**] Jens Carsten Jantzen, *Representations of algebraic groups*. Pure and Applied Mathematics, vol. 131. Academic Press Inc., Boston, MA, 1987.

[**K72**] Aleksandr A. Kirillov, *Èlementy teorii predstavleniĭ*. Izdat. "Nauka", Moscow, 1972. (Translation: *Elements of the theory of representations*, Springer-Verlag, Berlin, Heidelberg, 1976.)

[**K61**] Shoshichi Kobayashi, *On automorphism groups of homogeneous complex manifolds*. Proc. Amer. Math. Soc. **12** (1961) 359–361.

[**K70**] Bertram Kostant, *Quantization and unitary representations. I. Prequantization*. Lecture Notes in Math. **170** (1970) 87–208.

[**K62**] Ray A. Kunze, *On the irreducibility of certain multiplier representations*. Bull. Amer. Math. Soc. **68** (1962) 93–94.

[**S54**] Jean-Pierre Serre, *Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d'après Armand Borel et André Weil)*. In: Séminaire Bourbaki, Vol. 2, Exp. No. 100, 1–8. Secrétariat mathématique, Paris, 1954.

[**S70**] Jean-Marie Souriau, *Structure des systèmes dynamiques*. Dunod, Paris, 1970. (Reprint: Éditions Jacques Gabay, Sceaux, 2008. Translation: *Structure of dynamical systems: A symplectic view of physics*. Birkhäuser, Boston, 1997.)

[**S51**] Norman E. Steenrod, *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[**S68**] Robert Steinberg, *Lectures on Chevalley groups*. Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[**S74**] Robert Steinberg, *Conjugacy classes in algebraic groups*. Lecture Notes in Mathematics, Vol. 366. Springer-Verlag, Berlin, 1974. Notes by Vinay V. Deodhar.

[**W25**] Hermann Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen. I*. Math. Z. **23** (1925) 271–309. *II, III*. Math. Z. **24** (1926) 328–395. *Nachtrag*. Math. Z. **24** (1926) 789–791.

[**Z96**] François Ziegler, *Quantum representations and the orbit method*. In: Méthode des Orbites et Représentations Quantiques, pp. 1–66. Ph. D. Thesis, Université de Provence, 1996.