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An Extremal Problem for Finite Lattices

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Abstract

For a positive integer k, a *k-square* S_k (more generally, a *square*) in $\mathbb{Z} \times \mathbb{Z}$ is any set $\{(i,j),(i+k,j),(i,j+k),(i+k,j+k)\} \subset \mathbb{Z} \times \mathbb{Z}$. Let $\boldsymbol{S}_k$ denote the class of k-squares $S_k \subset \mathbb{Z} \times \mathbb{Z}$. A set $A \subset \mathbb{Z} \times \mathbb{Z}$ is said to be *$\boldsymbol{S}_k$-free* if, for each $S_k \in \boldsymbol{S}_k$, we have that $S_k \not\subseteq A$. For positive integers M and N, let $L_{M,N} = [0, M-1] \times [0, N-1]$ be the $M \times N$ non-negative integer lattice. For positive integers $k_1, \ldots, k_\ell$, set

$$\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}\}) = \max\{|A| : A \subseteq L_{M,N} \text{ is } \boldsymbol{S}_{k_i}\text{-free for all } 1 \le i \le \ell\},$$

and when $\{\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}\} = \{\boldsymbol{S}_k\}$, we abbreviate this parameter to $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$.

Our first result gives an exact formula for $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$ for all integers $k, M, N \ge 1$, where $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) = (3/4 + o(1))MN$ holds for fixed k and $\min\{M,N\} \to \infty$. Our second result identifies a set $A_0 \subset L_{M,N}$ of size $|A_0| \ge (2/3)MN$ with the property that, for any integer $k \not\equiv 0 \pmod 3$, A_0 is $\boldsymbol{S}_k$-free. Our third result shows that $|A_0|$ is asymptotically best possible, in that for all integers $M, N \ge 1$, $\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) \le (2/3)MN + O(M+N)$. When $M = 3m$ is divisible by three, our estimates on the error $O(M+N)$ render exact formulas for $\mathrm{ex}(L_{3m,3}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\})$ and $\mathrm{ex}(L_{3m,6}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\})$.

1 Introduction

We consider an extremal problem on finite lattices. For a positive integer k, a *k-square* S_k (more generally, a *square*) in $\mathbb{Z} \times \mathbb{Z}$ is any set $\{(i,j),(i+k,j),(i,j+k),(i+k,j+k)\} \subset \mathbb{Z} \times \mathbb{Z}$. When $k = 1$, we call a 1-square a *unit square*, and when $k = 2$, we call a 2-square a *bi-unit square*. Let $\boldsymbol{S}_k$ denote the class of all k-squares $S_k \subset \mathbb{Z} \times \mathbb{Z}$. A set $A \subset \mathbb{Z} \times \mathbb{Z}$ is said to be *$\boldsymbol{S}_k$-free* if, for each $S_k \in \boldsymbol{S}_k$, we have that $S_k \not\subseteq A$. Now, for positive integers M and N, consider the $M \times N$ non-negative integer lattice $L_{M,N} = [0, M-1] \times [0, N-1]$. For positive integers $k_1, \ldots, k_\ell$, let

$$\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}\}) = \max\{|A| : A \subseteq L_{M,N} \text{ is } \boldsymbol{S}_{k_i}\text{-free for all } 1 \le i \le \ell\}$$

denote the *extremal number* for the simultaneous avoidance of $\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}$ in $L_{M,N}$. When $\ell = 1$ and $k_1 = k$, we write this parameter as $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$.

Our first result gives an exact formula for $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$, for all integers $M, N, k \ge 1$.

Theorem 1.1. *For all integers $M, N, k \ge 1$, where $r_M = M \pmod{2k}$ and $r_N = N \pmod{2k}$,*

$$\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) = MN - \left(\frac{M - r_M}{2} + \max\{r_M - k, 0\}\right)\left(\frac{N - r_N}{2} + \max\{r_N - k, 0\}\right).$$

We prove Theorem 1.1 in Section 2. Note that Theorem 1.1 gives $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_1) = MN - \lfloor M/2 \rfloor \lfloor N/2 \rfloor$, for all integers $M, N \ge 1$, and $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) = (3/4)MN$, whenever M and N are divisible by $2k$.

Our next result concerns the parameter $\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}\})$ when $\ell \ge 2$.

Theorem 1.2. *For all integers $M, N \ge 1$, there exists $A_0 \subset L_{M,N}$ of size $|A_0| = \lfloor (2/3)MN \rfloor + 1$ with the property that for all positive integers $k \not\equiv 0 \pmod 3$, A_0 is $\boldsymbol{S}_k$-free.*

Remark 1.1. In some cases, Theorem 1.2 can be slightly improved: if $M \equiv N \pmod 3$, where $M \neq N$ or $1 \neq M \equiv 1 \pmod 3$, then $|A_0| = \lfloor (2/3)MN \rfloor + 2$. We prove this, and Theorem 1.2, in Section 3. □

Theorem 1.2 implies that for all positive integers $k_1, \ldots, k_\ell \not\equiv 0 \pmod 3$, we have $\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_{k_1}, \ldots, \boldsymbol{S}_{k_\ell}\}) \geq (2/3)MN$. Our final result shows that this bound is asymptotically best possible.

Theorem 1.3. *Let integers $M, N \geq 1$ be given, where $N = 3n$ is divisible by three. Then,*

$$\mathrm{ex}(L_{M,3n}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) \leq \frac{2}{3}MN + \begin{cases} n & \text{if } n \equiv 0 \pmod 2, \\ n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Consequently, for all integers $M, N \geq 1$, $\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) \leq (2/3)MN + O(M+N)$.

Remark 1.2. Theorem 1.3 is best possible when $n = 1, 2$ and $M = 3m > N$ is divisible by three. Indeed, then Theorem 1.2, Remark 1.1, and Theorem 1.3 combine to give the formulas $\mathrm{ex}(L_{3m,3}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) = 6m + 2$ and $\mathrm{ex}(L_{3m,6}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) = 12m + 2$. We have recently learned from [1] (in progress) that a linear term in Theorem 1.3 is, to some extent, necessary. There, it was shown that $\mathrm{ex}(L_{M,N}, \{\boldsymbol{S}_1, \boldsymbol{S}_2\}) \geq (2/3)MN + (2/27)N$ holds for (at least) infinitely many pairs of integers $M \geq N \geq 1$. □

The paper is organized as follows. We prove Theorem 1.1 in Section 2. We prove Theorem 1.2 in Section 3. We prove Theorem 1.3 in Section 4. We conclude the Introduction with the following remark.

Remark 1.3. The problem of forbidding fixed squares in $L_{M,N}$ bears some resemblance to a case of the classical problem of Zarankiewicz [4]. In the language of $M \times N$ lattices, one seeks the maximum size $z(M,N) = |A|$ of a subset $A \subset L_{M,N}$ which forbids an arbitrary *rectangle* $\{(a,c), (a,d), (b,c), (b,d)\} \subseteq L_{M,N}$. It is known from the work of Kővári, Sós, and Turán [2] that $z(M,N) < (N-1)M^{1/2} + 2M$, and a (projective plane) construction of Reiman [3] shows $z(N,N) = N^{3/2}(1 + o(1))$. The maximum size $|A|$ of a subset $A \subset L_{M,N}$ which forbids an arbitrary square is discussed in [1]. □

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2 Proof of Theorem 1.1

Fix integers $M, N > k \geq 1$. For an integer $t \in \mathbb{Z}$, write $t_{2k} = t \pmod{2k}$, where $0 \leq t_{2k} < 2k$. Set

$$B_k = \{(i,j) \in L_{M,N} : i_{2k} \geq k \text{ and } j_{2k} \geq k\} \qquad \text{and} \qquad A_k = L_{M,N} \setminus B_k, \tag{1}$$

where Figure 1 gives a visual example of $B_3 \subset L_{13,11}$. We prove that $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) = |A_k|$, which (if true) implies the formula for $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$ promised by Theorem 1.1. Indeed,

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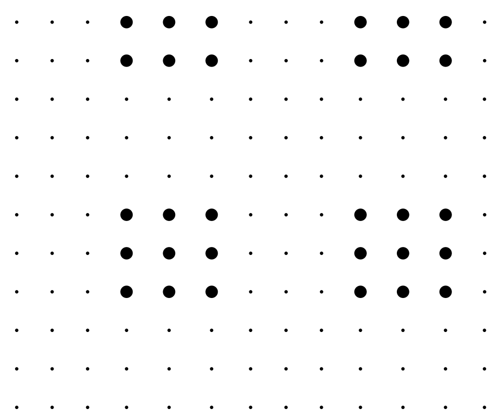

Figure 1: The set $B_3 \subset L_{13,11}$.

let $q_M, q_N \in \mathbb{N}$ and $0 \le r_M < 2k$ and $0 \le r_N < 2k$ satisfy $M = (2k)q_M + r_M$ and $N = (2k)q_N + r_N$. Then, it is easy to see that

$$|B_k| = (kq_M + \max\{r_M - k, 0\})\,(kq_N + \max\{r_N - k, 0\})\,,$$

which implies the formula for $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) = |A_k| = MN - |B_k|$ promised by Theorem 1.1.

To prove that $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) \ge |A_k|$, we show that A_k contains no k-squares. Indeed, assume, on the contrary, that $\{(i,j),(i+k,j),(i,j+k),(i+k,j+k)\} \subset A_k$, and w.l.o.g., let $i_{2k} < k$. If $j_{2k} < k$, then $(i+k,j+k) \in B_k$, and if $j_{2k} \ge k$, then $(i+k,j) \in B_k$, which is a contradiction either way.

To prove that $\mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) \le |A_k|$, let $A \subseteq L_{M,N}$ be given satisfying that

(a) A contains no k-squares;

(b) $|A| = \mathrm{ex}(L_{M,N}, \boldsymbol{S}_k)$.

Without loss of generality, assume that A is chosen to additionally satisfy that

(c) $|A \cap B_k|$ is a minimum.

We will show that $A \cap B_k = \emptyset$ (see Figure 1) so that (1) gives $A \subseteq A_k$, and hence $|A| \le |A_k|$.

Assume, on the contrary, that $A \cap B_k \ne \emptyset$, and let $(i,j) \in A \cap B_k$ be the minimum w.r.t. the lexicographic order on $L_{M,N}$. Since $(i,j) \in B_k$, we have from (1) that $i \ge i_{2k} \ge k$ and $j \ge j_{2k} \ge k$, so consider the points $(i-k,j),(i,j-k),(i-k,j-k) \in L_{M,N}$. Since A contains no k-squares, not all of these points can belong to A. We now consider these possibilities in cases (whose arguments are similar).

Case 1: $(i-k,j) \notin A$**.**

We claim that $A^* = (A \setminus \{(i,j)\}) \cup \{(i-k,j)\}$ contains no k-squares S_k, which contradicts Conditions (a)–(c) (since $|A^*| = |A|$ and $|A^* \cap B_k| < |A \cap B_k|$). Indeed, assuming otherwise, a k-square S_k in A^* must contain $(i-k,j)$. Since $(i,j) \notin A^*$, we must have $(i-2k,j) \in A$, and so $(i-2k,j) \in A \cap B_k$ contradicts our choice of (i,j).

Case 2: $(i, j-k) \notin A$**.**

Now, $A^* = (A \setminus \{(i,j)\}) \cup \{(i, j-k)\}$ contains no k-squares, since otherwise a k-square S_k in A^* contains $(i, j-k)$, and so $(i, j-2k) \in A \cap B_k$ contradicts our choice of (i,j).

Case 3: $(i-k, j-k) \notin A$**.**

Now, $A^* = (A \setminus \{(i,j)\}) \cup \{(i-k, j-k)\}$ contains no k-squares, since otherwise a k-square S_k in A^* contains $(i-k, j-k)$ and at least one of $(i-2k, j), (i-2k, j-2k), (i, j-2k) \in A \cap B_k$, which contradicts our choice of (i,j). □

Remark 2.1. A similar proof shows $B_k \subset L_{M,N}$ is a smallest set meeting all k-squares of $L_{M,N}$. Indeed, clearly B_k meets all k-squares of $L_{M,N}$, so let $B \subset L_{M,N}$ also do so, where $|B| = MN - \mathrm{ex}(L_{M,N}, \boldsymbol{S}_k) \leq |B_k|$. W.l.o.g., choose B so that $|B_k \setminus B|$ is a minimum. Assume $B_k \setminus B \neq \emptyset$, and let $(i,j) \in B_k \setminus B$ be the minimum w.r.t. the lexicographic order on $L_{M,N}$. Thus, for each $\boldsymbol{x} \in \{(i-2k, j), (i, j-2k), (i-2k, j-2k) \in \mathbb{Z} \times \mathbb{Z}$, if $\boldsymbol{x} \in L_{M,N}$, then $\boldsymbol{x} \in B_k \cap B$. Now, there exists $\boldsymbol{y} \in \{(i-k, j), (i, j-k), (i-k, j-k)\}$ so that $\boldsymbol{y} \in B \setminus B_k$. Define B^* by replacing $\boldsymbol{y} \in B$ with $(i,j) \in B_k$, but leaving all other memberships intact. Then, B^* meets all k-squares of $L_{M,N}$, while $|B^*| = |B|$ and $|B_k \setminus B^*| < |B_k \setminus B|$. □

3 Proof of Theorem 1.2

Fix integers $M, N \geq 1$. Define $A = \{(i,j) \in L_{M,N} : i \not\equiv j \pmod 3\}$ and $A_0 = A \cup \{(0,0)\}$. Using case-analysis on $MN \pmod 3$, one may show that $|L_{M,N} \setminus A| = \lceil MN/3 \rceil$. As such, $|A| = \lfloor 2MN/3 \rfloor$ and $|A_0| = \lfloor 2MN/3 \rfloor + 1$. For a fixed positive integer $k \not\equiv 0 \pmod 3$, we show A_0 contains no k-squares. Indeed, assume that $S_k = \{(i,j), (i+k, j), (i, j+k), (i+k, j+k)\} \subset A_0$, for some $(i,j) \in L_{M,N}$. If $i = j = 0$, then $(k,k) \in A$, which is impossible. We therefore assume that $(i,j) \neq (0,0)$ so that $S_k \subset A$, and we consider the cases $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$.

Case 1: $k \equiv 1 \pmod 3$**.**

Since $(i,j) \in A$, we have $i \equiv j+1 \pmod 3$ or $i \equiv j+2 \pmod 3$. If $i \equiv j+1 \pmod 3$, then $i \equiv j+k \pmod 3$, in which case $(i, j+k) \notin A$, a contradiction. If $i \equiv j+2 \pmod 3$, then $i+k \equiv i+1 \equiv j \pmod 3$, in which case $(i+k, j) \notin A$, a contradiction.

Case 2: $k \equiv 2 \pmod 3$**.**

Again, $i \equiv j+1 \pmod 3$ or $i \equiv j+2 \pmod 3$. If $i \equiv j+1 \pmod 3$, then $i+k \equiv i+2 \equiv j \pmod 3$, in which case $(i+k, j) \notin A$, a contradiction. If $i \equiv j+2 \pmod 3$, then $i \equiv j+k \pmod 3$, in which case $(i, j+k) \notin A$, a contradiction proving Theorem 1.2. □

We now argue the assertion of Remark 1.1. Assume $M \equiv N \pmod 3$, where $M \neq N$ or $1 \neq M \equiv 1 \pmod 3$. Define $\hat{A}_0 = A_0 \cup \{(M-1, N-1)\}$. Note that $(M-1, N-1) \notin A_0$ because $M \equiv N \pmod 3$ and $(M,N) \neq (1,1)$. For $k \not\equiv 0 \pmod 3$, we claim $\hat{A}_0$ contains no k-squares. Indeed, if $\hat{A}_0$ contains a k-square S_k, then $(M-1, N-1) \in S_k$, because A_0 contains no k-squares. Now, $(M-1-k, N-1-k) \in \hat{A}_0$. Since $M \equiv N \pmod 3$, it must be that $k = M-1 = N-1$, contradicting $M \neq N$ or $M \equiv 1 \pmod 3$.

4 Proof of Theorem 1.3

Let integers $M, N \geq 1$ be given, where $N = 3n$ is divisible by three, and let $A \subseteq L_{M,N}$ be a given subset containing neither unit nor bi-unit squares. Our goal is to show that

$$|A| \leq \frac{2}{3}MN + \begin{cases} n & \text{if } n \equiv 0 \pmod 2, \\ n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2}$$

In Section 4.1, we prove (2) when $n = 1$. In Section 4.2, we use the case $n = 1$ to prove (2) in general.

4.1 Proof of (2) when $n = 1$

For each $(i, j) \in L_{M,3}$, define

$$a_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in A, \\ 0 & \text{if } (i,j) \notin A, \end{cases} \qquad \boldsymbol{a}_i = \begin{bmatrix} a_{i,2} \\ a_{i,1} \\ a_{i,0} \end{bmatrix}, \qquad \text{and} \qquad \mathcal{A} = \{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{M-1}\}, \tag{3}$$

so that $\mathcal{A}$ is the set of incidence columns of A. For $0 \leq s \leq 3$, let $\mathcal{A}^{(s)} = \{\boldsymbol{a}_i \in \mathcal{A} : ||\boldsymbol{a}_i||^2 = s\}$, and define a characteristic function $\chi_s : \mathcal{A} \to \{0, 1\}$ by $\chi_s^{-1}(1) = \mathcal{A}^{(s)}$. We prove the following fact momentarily.

Fact 4.1. $|\mathcal{A}^{(3)}| \leq |\mathcal{A}^{(0)}| + |\mathcal{A}^{(1)}| + \chi_3(\boldsymbol{a}_0)\chi_2(\boldsymbol{a}_1) + \chi_2(\boldsymbol{a}_{M-2})\chi_3(\boldsymbol{a}_{M-1})$.

Fact 4.1 quickly implies (2) when $n = 1$. Indeed, we have $M = |\mathcal{A}| = \sum_{s=0}^{3} |\mathcal{A}^{(s)}|$, and so $|\mathcal{A}^{(2)}| = M - |\mathcal{A}^{(0)}| - |\mathcal{A}^{(1)}| - |\mathcal{A}^{(3)}|$. Therefore,

$$\begin{aligned} |A| = \sum_{i=0}^{M-1} ||\boldsymbol{a}_i||^2 = \sum_{s=0}^{3} s|\mathcal{A}^{(s)}| &= 2M + |\mathcal{A}^{(3)}| - |\mathcal{A}^{(1)}| - 2|\mathcal{A}^{(0)}| \\ &\overset{\text{Fact4.1}}{\leq} 2M + \chi_3(\boldsymbol{a}_0)\chi_2(\boldsymbol{a}_1) + \chi_2(\boldsymbol{a}_{M-2})\chi_3(\boldsymbol{a}_{M-1}), \end{aligned} \tag{4}$$

which implies (2) when $n = 1$. It remains to prove Fact 4.1. □

Proof of Fact 4.1. We begin with a couple elementary observations for a fixed $1 \leq i \leq M-2$. First, $||\boldsymbol{a}_{i-1}||^2 \leq 2$ or $||\boldsymbol{a}_{i+1}||^2 \leq 2$, since otherwise A contains a bi-unit square. Moreover, if $||\boldsymbol{a}_i||^2 = 3$, then $||\boldsymbol{a}_{i-1}||^2 \leq 1$ or $||\boldsymbol{a}_{i+1}||^2 \leq 1$, since otherwise an easy case analysis reveals that A contains a unit or a bi-unit square.

Now, define the graph $G = (V = \mathcal{A}, E)$ by putting, for each $\boldsymbol{a}_i \neq \boldsymbol{a}_j \in V = \mathcal{A}$, $\{\boldsymbol{a}_i, \boldsymbol{a}_j\} \in E$ if, and only if, $\boldsymbol{a}_i \in \mathcal{A}^{(3)}$, $\boldsymbol{a}_j \in \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}$, and $|i - j| = 1$. Since every edge of G has exactly one endpoint in $\mathcal{A}^{(3)}$, we have $\sum_{\boldsymbol{a}_i \in \mathcal{A}^{(3)}} \deg_G(\boldsymbol{a}_i) = \sum_{\boldsymbol{a}_j \in \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}} \deg_G(\boldsymbol{a}_j)$. By the preceding observations, all $\boldsymbol{a}_j \in \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}$ have $\deg_G(\boldsymbol{a}_j) \leq 1$, and all $\boldsymbol{a}_i \in \mathcal{A}^{(3)}$, except possibly $\boldsymbol{a}_0$ and $\boldsymbol{a}_{M-1}$ (if they belong to $\mathcal{A}^{(3)}$), have $\deg_G(\boldsymbol{a}_i) \geq 1$. Thus,

$$\begin{aligned} |\mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}| &\geq \sum_{\boldsymbol{a}_j \in \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}} \deg_G(\boldsymbol{a}_j) = \sum_{\boldsymbol{a}_i \in \mathcal{A}^{(3)}} \deg_G(\boldsymbol{a}_i) \\ &\geq |\mathcal{A}^{(3)} \setminus \{\boldsymbol{a}_0, \boldsymbol{a}_{M-1}\}| + \chi_3(\boldsymbol{a}_0) \deg_G(\boldsymbol{a}_0) + \chi_3(\boldsymbol{a}_{M-1}) \deg_G(\boldsymbol{a}_{M-1}) \\ &= |\mathcal{A}^{(3)}| - \chi_3(\boldsymbol{a}_0) \left(1 - \deg_G(\boldsymbol{a}_0)\right) - \chi_3(\boldsymbol{a}_{M-1}) \left(1 - \deg_G(\boldsymbol{a}_{M-1})\right). \end{aligned}$$

Fact 4.1 now follows. Indeed, assume, e.g., that $\chi_3(\boldsymbol{a}_0) = 1$, and observe then $\chi_2(\boldsymbol{a}_1) = 1 - \deg_G(\boldsymbol{a}_0)$. For $\chi_3(\boldsymbol{a}_0) = 1$ gives $\boldsymbol{a}_1 \notin \mathcal{A}^{(3)}$, and so $\deg_G(\boldsymbol{a}_0) = 1$, iff $\boldsymbol{a}_1 \in \mathcal{A}^{(0)} \cup \mathcal{A}^{(1)}$, iff $\boldsymbol{a}_1 \notin \mathcal{A}^{(2)}$, iff $\chi_2(\boldsymbol{a}_1) = 0$. □

4.2 Proof of (2)

We use (4) to prove (2). For integers $0 \leq j \leq n-1$ and $0 \leq s \leq 3$, define $L_{M,3,j} = [0, M-1] \times [3j, 3j+2]$ and $A_j = A \cap L_{M,3,j}$, and as in (3), define $\mathcal{A}_j = \{\boldsymbol{a}_{0,j}, \boldsymbol{a}_{1,j}, \ldots, \boldsymbol{a}_{M-1,j}\}$, $\mathcal{A}_j^{(s)} \subseteq \mathcal{A}_j$, and $\chi_{s,j} : \mathcal{A}_j \to \{0,1\}$. Observe that (4) implies that for each $0 \leq j \leq n-2$,

$$|A_j| = 2M + 2 \implies |A_{j+1}| \leq 2M \quad \text{and} \quad |A_{j+1}| = 2M + 2 \implies |A_j| \leq 2M. \tag{5}$$

To see (5), assume, e.g., that $|A_0| = 2M + 2$ and $|A_1| \geq 2M + 1$. By (4), $\chi_{3,0}(\boldsymbol{a}_{0,0}) = \chi_{2,0}(\boldsymbol{a}_{1,0}) = 1$, and w.l.o.g., $\chi_{3,1}(\boldsymbol{a}_{0,1}) = \chi_{2,1}(\boldsymbol{a}_{1,1}) = 1$. Then, $\boldsymbol{a}_{0,0} = \boldsymbol{a}_{0,1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, and since A_0, A_1 each contain no unit squares, $\boldsymbol{a}_{1,0} = \boldsymbol{a}_{1,1} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$. Now, we have that $\{(0,2), (1,2), (0,3), (1,3)\} \subset A$, a contradiction.

To prove (2), call an index $0 \leq j \leq n-1$ *big* (*medium*) ((*small*)) if $|A_j| = 2M + 2$ ($|A_j| = 2M + 1$) (($|A_j| \leq 2M$)). Let β (μ) ((σ)) be the number of big (medium) ((small)) indices $0 \leq j \leq n-1$. On the one hand, $\sigma + \mu + \beta = n$, and so $\mu = n - \sigma - \beta$. On the other hand, we have

$$\begin{aligned} |A| = \sum_{j=0}^{n-1} |A_j| &= \sum_{j \text{ is big}} |A_j| + \sum_{j \text{ is medium}} |A_j| + \sum_{j \text{ is small}} |A_j| \leq (2M+2)\beta + (2M+1)\mu + 2M\sigma \\ &= 2M(\sigma + \mu + \beta) + \mu + 2\beta = 2Mn + n + \beta - \sigma = \frac{2}{3}MN + n + \beta - \sigma. \end{aligned}$$

Observe that $\beta \leq \sigma + 1$, and when n is even, $\beta \leq \sigma$ (which implies (2)). Indeed, let $0 \leq j_1 < \cdots < j_\beta \leq n-1$ be the big indices. Then (5) implies that $\sigma \geq \beta - 1$, since between every consecutive pair $j_\ell < j_{\ell+1}$ is at least one small index. More strongly, $\sigma \geq \beta$ holds if $j_1 \geq 1$, or $j_\beta \leq n-2$, or $j_{\ell+1} \geq j_\ell + 3$, for some $1 \leq \ell < \beta$. Otherwise, $j_1 = 0$, $j_2 = 2$, $j_3 = 4$, $\ldots$, $2(\beta - 1) = j_\beta = n - 1$, and so n is odd. □

References

[1] Gegner, E., Goldwasser, J., Noland, W., and Winslow, R. Maximum subsets of the integer lattice avoiding two rectangles. *in preparation*, 2015.

[2] Kővári, T., Sós, V., and Turán, P. On a problem of K. Zarankiewicz. *Colloquium Math.*, **3**: 50–57, 1954.

[3] Reiman, I. Über ein Problem von K. Zarankiewicz. *Acta Mathematica Academiae Scientiarum Hungariacae*, **9**: 269–273, 1958.

[4] Zarankiewicz, K. Problem P 101. *Colloq. Math.*, **2**: 301, 1951.

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